On bayesian bonus-malus premium under linex loss function with applications

Sobre o prémio de bónus-malus bayesiano sob a função de perda linex com aplicações

Sobre la prima bayesiana bonus-malus bajo la función de pérdida linex con aplicaciones

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ABSTRACT

The majority of bonus-malus systems (BMS) determine the premium for each policyholder solely based on their claim frequency. Adopting this method would be unjust. A driver who files a claim for small damage cannot be subjected to the same penalties as a driver responsible for a claim involving significant damage. To
achieve a balance in the portfolio between good and bad drivers, a new portion of the premium is computed as a compromise. To do this, the claim severity is created and assessed to provide a fair premium. The consideration was determined by taking into account both the quantity and magnitude of the claims. This work assumes that the frequency of claims follows a Poisson-Akash distribution, but the severity of claims follows a newly proposed distribution known as the Inverse-Gamma Lindley distribution. The bonus-malus premiums were computed using the Bayesian approach for both the frequency and severity of claims. The premium is computed using the asymmetric Linex loss function for both the frequency and severity of claims in the bonus-malus system. Illustrative instances utilizing an actual dataset for the application of BMS are presented, showcasing the use of claim frequency alone and the combination of claim frequency and claim severity. The R software is employed for these demonstrations. We concluded this paper with a discussion comparing the premiums obtained in each case.

**Keywords:** Bonus-Malus System, Poisson-Akash Distribution, Inverse-Gamma Lindley Distribution, Bayesian Premium, Linex Loss Function.

**RESUMO**
A maioria dos sistemas de bonus-malus (BMS) determina o prémio de cada tomador de seguro apenas com base na sua frequência de sinistros. A adoção deste método seria injusta. Um condutor que apresenta um sinistro por danos de pequena monta não pode ser objeto das mesmas sanções que um condutor responsável por um sinistro com danos importantes. Para conseguir um equilíbrio na carteira entre os bons e os maus condutores, é calculada uma nova parte do prémio como compromisso. Para tal, a gravidade do sinistro é criada e avaliada de modo a proporcionar um prémio justo. A consideração foi determinada tendo em conta tanto a quantidade como a magnitude dos sinistros. Este trabalho assume que a frequência dos sinistros segue uma distribuição Poisson-Akash, mas a gravidade dos sinistros segue uma distribuição recentemente proposta, conhecida como distribuição Inverse-Gamma Lindley. Os prémios de bónus-malus foram calculados utilizando a abordagem Bayesiana, tanto para a frequência como para a gravidade dos sinistros. O prémio é calculado utilizando a função de perda Linex assimétrica tanto para a frequência como para a gravidade dos sinistros no sistema de bónus-malus. São apresentados exemplos ilustrativos utilizando um conjunto de dados reais para a aplicação do BMS, demonstrando a utilização apenas da frequência dos sinistros e a combinación da frequência e da gravidade dos sinistros. O software R é utilizado para estas demonstrações. Concluímos este artigo com uma discussão que compara os prémios obtidos em cada caso.

**Palavras-chave:** Sistema de Bónus-Malus, Distribuição Poisson-Akash, Distribuição Lindley Gama Inversa, Prémio Bayesiano, Função de Perda Linex.

**RESUMEN**
La mayoría de los sistemas de bonus-malus (BMS) determinan la prima de cada asegurado únicamente en función de su frecuencia de siniestralidad. Adoptar este método sería injusto. Un conductor que presenta un siniestro por pequeños daños no puede ser objeto de las mismas penalizaciones que un conductor responsable de un siniestro con daños importantes. Para lograr un equilibrio en la cartera entre
buenos y malos conductores, se computa una nueva parte de la prima como compromiso. Para ello, se crea y evalúa la gravedad del siniestro para obtener una prima justa. La consideración se determina teniendo en cuenta tanto la cantidad como la magnitud de los siniestros. Este trabajo parte de la base de que la frecuencia de los siniestros sigue una distribución Poisson-Akash, pero la gravedad de los siniestros sigue una distribución propuesta recientemente conocida como distribución Inverse-Gamma Lindley. Las primas de bonus-malus se calcularon utilizando el enfoque bayesiano tanto para la frecuencia como para la gravedad de los siniestros. La prima se calcula utilizando la función de pérdida asimétrica de Linex tanto para la frecuencia como para la gravedad de los siniestros en el sistema bonus-malus. Se presentan casos ilustrativos utilizando un conjunto de datos reales para la aplicación de BMS, mostrando el uso de la frecuencia de siniestros por sí sola y la combinación de la frecuencia de siniestros y la gravedad de los mismos. Para estas demostraciones se emplea el software R. Concluimos este artículo con una discusión en la que se comparan las primas obtenidas en cada caso.

Palabras clave: Sistema Bonus-Malus, Distribución Poisson-Akash, Distribución Inverse-Gamma Lindley, Prima Bayesiana, Función de Pérdida Linex.

1 INTRODUCTION

Insurance is a legally binding agreement between two individuals or entities, namely the insurer and the insured. The insurer agrees to provide compensation in the event of any harm or loss. As part of this policy, the insured is obligated to pay a premium, as specified in the contract. Automobile insurance is the primary domain of the insurance industry. The bonus-malus method is frequently utilized in automotive insurance to adjust the premium that the policyholder must pay according to their claim history (Lemaire, 1995). The primary goal of the bonus-malus system is to reward the responsible driver for a year without any accidents. Conversely, an incompetent driver will face a penalty (malus) in the form of an increased insurance price for the next year. The objective is to establish an experience rating system that considers both the individual’s and the group’s combined experiences to determine the premium for the upcoming year. The credibility hypothesis is employed to forecast the anticipated claims experience of an individual risk within a diverse collective, where the risks are not uniform. The primary objective is to ascertain the appropriate weighting to assign to each distinct risk dataset to calculate a justifiable premium to be levied. For comprehensive and up-to-date overviews of credibility theory, see Norberg (2004), Buhlmann and
The Poisson distribution is commonly used to describe the number of claims in automobile insurance. Nevertheless, a basic Poisson distribution falls short of accurately describing the number of claims in an insurance portfolio. For that, Lemaire (1995) demonstrated that a mixed Poisson distribution has a more pronounced tail, leading to improved outcomes for claim frequency in an optimal bonus-malus system, see Tremblay (1992). However, it would be unjust to determine the insurance price only primarily on the number of claims made by the insured. A driver who filed a claim for little damages should not be subjected to the same penalties as a driver who filed a claim for significant damages. To achieve a balanced portfolio between excellent and poor drivers, we introduce and calculate the claim severity in order to determine a new portion of the premium. This serves as a compromise to ensure that all drivers pay an equal premium (see Frangos and Vrontos, 2001; Moumeesri, 2020). This work also introduces the Varians (1995) asymmetric Linex loss function. We widely use this loss function to determine the Bayesian estimators and calculate the Bayesian premium. See Metiri (2016); Busu (2011); Sadoun (2017). This study will employ the Linex loss function to compute a fresh bonus-malus premium for the model that relies only on the claim frequency component, as well as for the model that takes into account both the frequency and severity of the claims. The subsequent sections of this work are structured in the following manner: Sections 2 and 3 provide a sequential description of the distribution of claim frequency and claim severity, respectively. The sections are composed of five subsections, namely: mixing distribution, maximum likelihood estimator, Bayesian technique, premium computation, and premium under the Linex loss function. Section 4 of the report focuses on the practical implementation of numerical analysis using R software. It utilizes actual claim data and presents the findings in three distinct subsections: claim frequency, claim severity using the Lognormal-Gamma distribution, and claim severity using the Inverse-Gamma Lindley distribution. Section 5 provides the ultimate findings of this study.

The objective of this article is to calculate the bonus malus premiums using the Poisson Akash distribution for the claim frequency and the Inverse Gamma Lindley distribution for the claim severity in order to have an equal and fair premium for each policyholder according to their claim frequency and the severity of each claim.
2 CLAIM FREQUENCY DISTRIBUTION USING POISSON-AKASH

2.1 MIXING DISTRIBUTION

l is the claim number; it is assumed to follow a Poisson distribution with parameter \( \theta \) and a probability mass function

\[
P(l|\theta) = \frac{e^{-\theta l}}{l!}, \quad l = 0, 1, 2, ..., l > 0.
\] (1)

The Poisson random variable has an expected value of

\[
E[l|\theta] = Var[l|\theta] = \theta.
\] (2)

We suppose that \( \theta \) follows the Akash distribution with parameter \( \gamma \). So, the probability density function (pdf) of \( \theta \) can be expressed as follows

\[
\pi(\theta) = \frac{\gamma^3 (1 + \theta^2)e^{-\gamma \theta}}{\gamma^2 + 2}.
\] (3)

Now, the representation of the Poisson mixed with Akash distribution (see Shanker (2016)) is

\[
f(l) = \int_0^\infty P(l|\theta) \pi(\theta) d\theta
\]

\[
= \int_0^\infty \frac{e^{-\theta l}}{l!} \left( \frac{\gamma^3}{\gamma^2 + 2} \right) (1 + \theta^2)e^{-\gamma \theta} d\theta
\]

\[
= \frac{\gamma^3}{\gamma^2 + 2} \left[ \int_0^\infty \theta^l e^{-\theta(1+\gamma)} d\theta + \int_0^\infty \theta^{l+2} e^{-\theta(1+\gamma)} d\theta \right]
\]

\[
= \frac{\gamma^3}{\gamma^2 + 2} \left[ \frac{\Gamma(l+1)}{(1+\gamma)^{l+1}} + \frac{\Gamma(l+3)}{(1+\gamma)^{l+3}} \right]
\]

\[
= \frac{\gamma^3}{\gamma^2 + 2} \left[ \frac{l!}{(1+\gamma)^{l+1}} + \frac{(l+2)(l+1)!}{(1+\gamma)^{l+3}} \right]
\]

\[
= \frac{\gamma^3}{\gamma^2 + 2} \left[ \frac{(1+\gamma)^2 + (l+2)(l+1)}{(1+\gamma)^{l+3}} \right]
\]

\[
f(l) = \frac{\gamma^3}{\gamma^2 + 2} \frac{\gamma^2 + 2\gamma + l^2 + 3l + 3}{(1+\gamma)^{l+3}}
\] (4)
2.2 MAXIMUM LIKELIHOOD ESTIMATOR

The Maximum Likelihood estimation (MLE) is commonly employed to estimate model parameters. The primary objective is to optimize the likelihood function. The concept of maximum likelihood involves estimating the value of a parameter that maximizes the likelihood of the observed data. \( l_1, l_2, \ldots, l_n \) is a random sample from Poisson-Akash distribution in (4). Where the sample size is \( n \).

To obtain the most likely value of the parameter \( \gamma \), the likelihood function \( L \) should be maximized as follows

\[
L(\gamma, l_i) = \prod_{i=1}^{n} f(l_i, \gamma) = \prod_{i=1}^{n} \frac{\gamma^3 l_i^2 + 3l_i + (\gamma^2 + 2\gamma + 3)}{(1 + \gamma)^{l_i + 3}}
\]

\[
\left( \frac{\gamma^3}{\gamma^2 + 2} \right)^n \prod_{i=1}^{n} \frac{l_i^2 + 3l_i + (\gamma^2 + 2\gamma + 3)}{(1 + \gamma)^{l_i + 3}}
\]

\[ (5) \]

The log-likelihood function is

\[
\ln L(\gamma, l_i) = n \ln \left( \frac{\gamma^3}{\gamma^2 + 2} \right) - \sum_{i=1}^{n} \ln(\gamma + 1)^{l_i + 3}
\]

\[ + \sum_{i=1}^{n} \ln[l_i^2 + 3l_i + (\gamma^2 + 2\gamma + 3)] \]

\[ = 3n \ln \gamma - n \ln(\gamma^2 + 2) - \ln(1 + \gamma) \sum_{i=1}^{n} (l_i + 3)
\]

\[ + \sum_{i=1}^{n} \ln(\gamma^2 + 2\gamma + l_i^2 + 3l_i + 3)
\]

\[ (6) \]

The solution of the equation \( \frac{\partial \ln L(\gamma, l_i)}{\partial \gamma} = 0 \), provides the estimator \( \hat{\gamma} \) of the parameter \( \gamma \). Therefore

\[
\frac{\partial \ln L(\gamma, l_i)}{\partial \gamma} = 0,
\]

\[
\frac{\partial}{\partial \gamma} [3n \ln \gamma - n \ln(\gamma^2 + 2) - \ln(1 + \gamma) \sum_{i=1}^{n} (l_i + 3)
\]

\[ + \sum_{i=1}^{n} \ln(\gamma^2 + 2\gamma + l_i^2 + 3l_i + 3)] = 0,
\]

\[
\frac{3n}{\gamma} - \frac{2n\gamma}{\gamma^2 + 2} - \sum_{i=1}^{n} \frac{2(l_i + 3)}{\gamma + 1} + \sum_{i=1}^{n} \frac{2(\gamma + 1)}{(\gamma^2 + 2\gamma + l_i^2 + 3l_i + 3)} = 0.
\]

\[ (7) \]
The estimator is not presented in its explicit form; rather, numerical solutions may be used to determine it.

2.3 BAYESIAN METHOD

\[ l_1, l_2, \ldots, l_t \] is a sample of size \( t \). \( N = \sum_{i=1}^{t} l_i \) is total number that a policyholder has made in \( t \) years, where \( l_i \) is the claim number made by a policyholder over \( i \) years, \( i = 1, 2, \ldots, t \).

The likelihood function is

\[
L(\theta; l_1, l_2, \ldots, l_n) = \prod_{i=1}^{t} \frac{e^{-\theta l_i}}{l_i!} = \frac{1}{\prod_{i=1}^{t} l_i!} e^{-\theta \sum l_i} \propto e^{-\theta \sum l_i N}. \tag{8}
\]

The Prior distribution is

\[
\pi(\theta) = \frac{\gamma^3}{\gamma + 2} (1 + \theta^2) e^{-\gamma \theta} \\
\propto (1 + \theta^2) e^{-\gamma \theta}. \tag{9}
\]

To calculate the posterior distribution function for a policyholder with claim history \( l_1, l_2, \ldots, l_t \), we utilize the Bayes theorem. The posterior distribution function is derived by multiplying the likelihood function and the prior distribution as follows

\[
\pi^*(\theta | l_1, \ldots, l_n) \propto P(l_1, \ldots, l_n | \theta) \pi(\theta) \\
= e^{-\theta N (1 + \theta^2)} e^{-\gamma \theta} \\
= \theta^N (1 + \theta^2) e^{-\theta (t + \gamma)}. \tag{10}
\]

Consider

\[
\int_0^\infty \pi^*(\theta | l_1, \ldots, l_n) d\theta \propto \int_0^\infty e^{-\theta (t + \gamma) (\theta^2 + 1)} \theta^N d\theta. \tag{11}
\]

Then
\[
\int_0^\infty A e^{-\theta(t+\gamma)}(\theta^2 + 1)\theta^N d\theta = 1,
\]

(12)

where \( A \) is a constant. It comes from

\[
A\left[\int_0^\infty \theta^{N+2} e^{-(t+\gamma)\theta} d\theta + \int_0^\infty \theta^N e^{-(t+\gamma)\theta} d\theta\right] = 1,
\]

\[
A \left[ \frac{\Gamma(n+3)}{(t+\gamma)^{n+3}} + \frac{\Gamma(n+1)}{(t+\gamma)^{n+2}} \right] = 1,
\]

\[
A \left[ \frac{(n+3)(t+\gamma)^N + (n+1)(t+\gamma)^{N-1}}{(t+\gamma)^{n+3}} \right] = 1.
\]

(13)

Then

\[
A = \frac{(t+\gamma)^{n+3}}{\Gamma(n+1)(n+2)(n+1)+(t+\gamma)^2}.
\]

(14)

Thus, the posterior distribution function for the claim frequency is expressed as

\[
\pi^*(\theta|l_1, \ldots, l_n) = \frac{(t+\gamma)^{n+3}}{\Gamma(n+1)(n+2)(n+1)+(t+\gamma)^2} \cdot \theta^N(1 + \theta^2)e^{-(t+\gamma)\theta}.
\]

(15)

### 2.4 PREMIUM COMPUTATION

In this study, we shall compute the net premium, which refers to the fundamental premium. The word is defined as the average number of claims per insured individual. If we have a policyholder’s claim history denoted as \( l_1, l_2, \ldots, l_t \), then the mean of the posterior distribution function for the Poisson-Akash distribution (which represents the predicted number of claims from this policyholder) may be calculated.
\[
\theta_{t+1} = E[\theta | l_1, ..., l_t] = E[l_1, ..., l_n | \theta]
\]

\[
= \int_0^\infty \theta \pi^*(\theta_i | l_1, ..., l_n)
\]

\[
\frac{(t+\gamma)^{n+3}}{\Gamma(n+1)[(n+2)(n+1)+(t+\gamma)^2]}
\]

\[
\times \left[ \frac{\Gamma(n+4)}{(t+\gamma)^{n+4}} + \frac{\Gamma(n+2)}{(t+\gamma)^{n+2}} \right]
\]

\[
= \frac{(t+\gamma)^{n+3}}{\Gamma(n+1)[(n+2)(n+1)+(t+\gamma)^2]}
\]

\[
\times \left[ \frac{\Gamma(n+4)+(t+\gamma)^2\Gamma(n+2)}{(t+\gamma)^{n+4}} \right]
\]

\[
\frac{(n+1)(n+2)(n+3)\Gamma(n+1)(t+\gamma)^2}{(t+\gamma)^3+(t+\gamma)(n+2)(n+1)}
\]

\[
= \frac{(n+1)(n+2)(n+3)+n(n+1)(t+\gamma)^2}{(t+\gamma)^3+(t+\gamma)(n+2)(n+1)}.
\]  \quad (16)

The initial premium at time \( t=0 \) is supposed to be 100. The premium at time \( t+1 \) is then

\[
Premium_{t+1} = 100 \frac{y^2+2}{y^2+6}
\]

\[
\times \frac{(n+1)(n+2)(n+3)+n(n+1)(t+\gamma)^2}{(t+\gamma)^3+(t+\gamma)(n+2)(n+1)}
\]  \quad (17)

2.5 PREMIUM UNDER LINEX LOSS FUNCTION

Within this subsection, we shall employ the asymmetric Linex loss function to compute the bonus-malus premiums. The name of this loss function is derived from its applicability to issues that exhibit linearity on one side and exponential behavior on the other, such as those encountered in statistical estimation and prediction analyses. See, for instance, Varians (1995), Rojo (1989) and Nassar (2004). This Linex loss function maybe expressed as

\[
L(\hat{\theta}, \theta) = \exp\left(a(\hat{\theta} - \theta)\right) - a(\hat{\theta} - \theta) - 1, \quad a \neq 0,
\]  \quad (18)
where:

\( \hat{\theta} \) is the estimator of \( \theta \) under the Linex loss function which minimizes the above equation (see Zellner, 1986), \( \hat{\theta} \) is given by

\[
\hat{\theta} = -\frac{1}{a} \ln \left( E \left[ e^{-a\theta} | X \right] \right). \tag{19}
\]

Then

\[
E \left[ e^{-a\theta} | l_1, ..., l_n \right] = \int_0^\infty e^{-a\theta} \pi^*(\theta_i | l_1, ..., l_n) d\theta
\]

\[
\propto \int_0^\infty \theta^{n+2} e^{-(t+\gamma+a)\theta} d\theta + \int_0^\infty \theta^n e^{-(t+\gamma+a)\theta} d\theta
\]

\[
= \left[ \frac{\Gamma(n+3)}{(t+\gamma+a)^{n+1}} + \frac{\Gamma(n+1)}{(t+\gamma+a)^{n+1}} \right]
\]

\[
\times \frac{(t+\gamma)^{n+3}}{\Gamma(n+1)(n+2)(n+1)+(t+\gamma)^2}
\]

\[
E \left[ e^{-a\theta} | l_1, ..., l_n \right] = \frac{(n+2)(n+1)+(t+\gamma+a)^2}{(t+\gamma+a)^{n+3}} \tag{20}
\]

Finally

\[
\hat{\theta} = -\frac{1}{a} \ln \left( \frac{(n+2)(n+1)+(t+\gamma+a)^2}{(t+\gamma+a)^{n+3}} \right) \tag{21}
\]

At time \( t=0 \), the initial premium is supposed to be 100. The premium at time \( t+1 \) is given as follows

\[
Premium_{t+1} = 100 \frac{y(y^2+2)}{y^2+6} \tag{22}
\]

\[
\times \left[ -\frac{1}{a} \ln \left( \frac{(n+2)(n+1)+(t+\gamma+a)^2}{(t+\gamma+a)^{n+3}} \right) \right]
\]

\[
\times \frac{(t+\gamma)^{n+3}}{(n+2)(n+1)+(t+\gamma)^2}.
\]
3 CLAIM SEVERITY DISTRIBUTION USING INVERSE-GAMMA LINDLEY

3.1 MIXING MODEL

$X$ is a random variable that represents the size of each policyholder’s claims. We suppose that $X$ follows an Inverse-Gamma distribution with a PDF given as

$$f(x|\lambda) = \frac{\lambda^{\alpha}x^{-\alpha-1}}{\Gamma(\alpha)} e^{-\frac{\lambda}{x}}.$$  \hspace{1cm} (23)

The Inverse-Gamma random variable has an expected value of

$$E[X] = \frac{\lambda}{\alpha-1}$$  \hspace{1cm} (24)

Suppose that $\lambda$ follows a Lindley distribution with parameter $\beta$. Then, the probability density function of $\lambda$ is given below

$$\pi(\lambda) = \frac{\beta^2}{\beta+1} (\lambda + 1)e^{-\beta \lambda}.$$  \hspace{1cm} (25)

Therefore, the mixed Inverse-Gamma with Lindley distribution is obtained as follows

$$f(x) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\lambda}{x}} \frac{\beta^2}{\beta+1} (\lambda + 1)e^{-\beta \lambda} d\lambda$$

$$= \frac{x^{-\alpha-1} \beta^2}{\Gamma(\alpha)(\beta+1)} \int_0^\infty \lambda^\alpha (\lambda + 1)e^{-\lambda(\beta + \frac{1}{\lambda})} d\lambda$$

$$= \frac{x^{-\alpha-1} \beta^2}{\Gamma(\alpha)(\beta+1)} \left[ \int_0^\infty \lambda^{\alpha+1} e^{-\lambda(\beta + \frac{1}{\lambda})} d\lambda + \int_0^\infty \lambda^\alpha e^{-\lambda(\beta + \frac{1}{\lambda})} d\lambda \right]$$

$$= \frac{x^{-\alpha-1} \beta^2}{\Gamma(\alpha)(\beta+1)} \left[ \Gamma(\alpha+2)(\beta + \frac{1}{\lambda})^{-\alpha-2} + \frac{\Gamma(\alpha+1)}{(\beta + \frac{1}{\lambda})^{-\alpha-1}} \right]$$

$$= \frac{x^{-\alpha-1} \beta^2 \Gamma(\alpha+1)(\beta + \frac{1}{\lambda})^{-\alpha-2}}{\Gamma(\alpha)(\beta+1)(\beta + \frac{1}{\lambda})^{-\alpha-1}} \left[ \alpha + 1 + \beta + \frac{1}{x} \right]$$

$$f(x) = \frac{x^{-\alpha-1} \beta^2 \Gamma(\alpha+1)}{(\beta+1)(\beta + \frac{1}{\lambda})^{-\alpha-1}} \left[ \alpha + 1 + \beta + \frac{1}{x} \right].$$  \hspace{1cm} (26)
3.2 MAXIMUM LIKELIHOOD ESTIMATOR

\( X = (X_1, X_2, \ldots, X_n)^T \) is a vector of an identically independent observation for the Inverse-Gamma Lindley distribution with probability density function in (26).

The most likely value of parameters \( \beta \) that maximizes the likelihood function is obtained as follows. The likelihood function is

\[
L(\beta, x_i) = \prod_{i=1}^{n} f(x_i, \beta) \\
= \prod_{i=1}^{n} \frac{x_i^{-\alpha-1} \beta^{2(\alpha+1)}}{(\beta+1)^{\alpha+2} x_i^2} \left[ \alpha + 1 + \beta + \frac{1}{x_i} \right]
\]

\[
L(\beta, x_i) = \frac{\beta^{2n(\alpha+1)}}{(\beta+1)^n} \prod_{i=1}^{n} \frac{x_i^{-\alpha-1}}{(\beta+1/x_i)^{\alpha+2}} \left[ \alpha + 1 + \beta + \frac{1}{x_i} \right]. 	ag{27}
\]

The Log-likelihood function is expressed as

\[
\ln L(\beta, x_i) = 2n \ln \beta + n \ln(\alpha + 1) - n \ln(\beta + 1) + \sum_{i=1}^{n} (\alpha + 1 - 1) \ln x_i \\
- (\alpha + 2) \sum_{i=1}^{n} \ln \left( \beta + \frac{1}{x_i} \right) + \sum_{i=1}^{n} \ln \left( \alpha + 1 + \beta + \frac{1}{x_i} \right) \\
= n \left[ - \ln(\beta + 1) + 2 \ln \beta + \ln(\alpha + 1) \right] - (\alpha + 1) \sum_{i=1}^{n} \ln x_i \\
- (\alpha + 2) \sum_{i=1}^{n} \ln \left( \beta + \frac{1}{x_i} \right) + \sum_{i=1}^{n} \ln \left( \alpha + 1 + \beta + \frac{1}{x_i} \right). 	ag{28}
\]

Therefore, the estimator of \( \hat{\beta} \) of the parameter \( \beta \) is

\[
\frac{\partial \ln L(\beta; x_i)}{\partial \beta} = 0,
\]

\[
n \left[ - \ln(\beta + 1) + 2 \ln \beta + \ln(\alpha + 1) \right] - (\alpha + 1) \sum_{i=1}^{n} \ln x_i \\
- (\alpha + 2) \sum_{i=1}^{n} \ln \left( \beta + \frac{1}{x_i} \right) + \sum_{i=1}^{n} \ln \left( \alpha + 1 + \beta + \frac{1}{x_i} \right) = 0.
\]

\[
\frac{2n}{\beta} - \frac{n}{\beta+1} - (\alpha + 2) \sum_{i=1}^{n} \left( \frac{1}{\beta+1/x_i} \right) + \sum_{i=1}^{n} \frac{1}{\alpha+1+\beta+1/x_i} = 0. 	ag{29}
\]

The estimator is not in its explicit form, numerical solutions can be applied to determine it.
3.3 ON BAYESIAN METHOD

\[ N = \sum_{i=1}^l l_i \] is the claim's total number made by an insurer over \( t \) years. Let \( X \) be the size of claim \( l \) for \( l = 1, 2, \ldots, N \). The likelihood function is

\[
L(\lambda|x_1,\ldots,x_n) = f(x_1,\ldots,x_n|\lambda) = \prod_{i=1}^n \frac{\lambda^a x_i^{-\alpha-1}}{\Gamma(\alpha)} e^{-\frac{\lambda}{x_i}} \\
= \frac{\lambda^n}{(\Gamma(\alpha))^n} \prod_{i=1}^n x_i^{-\alpha-1} e^{-\lambda \sum_{i=1}^n \frac{1}{x_i}} \\
\propto \lambda^n e^{-\lambda \sum_{i=1}^n \frac{1}{x_i}}
\]

(30)

The Prior distribution is

\[
\pi(\lambda) \propto (\lambda + 1)e^{-\beta \lambda}.
\]

(31)

The theorem of Bayes is applied to obtain the posterior distribution function as follows

\[
\pi^*(\lambda|x_1,\ldots,x_n) \propto f(x_1,\ldots,x_n|\lambda)\pi(\lambda) \\
\propto \lambda^n (1 + \lambda) e^{-\lambda (\beta + \sum_{i=1}^n \frac{1}{x_i})}.
\]

(32)

Consider

\[
\int_0^\infty \pi^*(\lambda|x_1,\ldots,x_n) d\lambda \propto \int_0^\infty \lambda^n (1 + \lambda) e^{-\lambda (\beta + \sum_{i=1}^n \frac{1}{x_i})} d\lambda.
\]

(33)

Then

\[
\int_0^\infty \pi^*(\lambda|x_1,\ldots,x_n) d\lambda = \int_0^\infty B\lambda^n (1 + \lambda) e^{-\lambda (\beta + \sum_{i=1}^n \frac{1}{x_i})} d\lambda = 1.
\]

(34)

where: \( B \) is a constant. It results in
Then

\[
B = \frac{\left(\beta + \sum_{i=1}^{n} \frac{1}{x_i}\right)^{an+2}}{\Gamma(an+1)(an+2+\beta + \sum_{i=1}^{n} \frac{1}{x_i})},
\]  
(36)

Thus, the posterior distribution function for the severity component is given by

\[
\pi^{*}(\lambda|x_1, ..., x_n) = \frac{\left(\beta + \sum_{i=1}^{n} \frac{1}{x_i}\right)^{an+2}}{\Gamma(an+1)(an+2+\beta + \sum_{i=1}^{n} \frac{1}{x_i})} \pi(\lambda) \exp\left(-\lambda \left(\beta + \sum_{i=1}^{n} \frac{1}{x_i}\right)\right).
\]  
(37)

\[
\lambda^{*+1} = E[\lambda|x_1, ..., x_n] = \int_{0}^{\infty} \lambda \pi^{*}(\lambda|x_1, ..., x_n) d\lambda
\]

\[
= \frac{\left(\beta + \sum_{i=1}^{n} \frac{1}{x_i}\right)^{an+2}}{\Gamma(an+1)(an+2+\beta + \sum_{i=1}^{n} \frac{1}{x_i})} \cdot \frac{\Gamma(an+2)}{\Gamma(an+1)(an+2+\beta + \sum_{i=1}^{n} \frac{1}{x_i})} + \frac{\Gamma(an+3)}{\Gamma(an+2+\beta + \sum_{i=1}^{n} \frac{1}{x_i})}
\]

\[
= \frac{(\Gamma(an+2))}{\Gamma(an+1)(an+2+\beta + \sum_{i=1}^{n} \frac{1}{x_i})}.
\]

3.4 AROUND PREMIUM CALCULATION

In this part, as in the claim frequency, we will use the net premium. The expected value of the precedent formula (37) for the Inverse-Gamma Lindley distribution is
\[
\lambda_{t+1} = \frac{(\alpha n + 1)(\alpha n + 2 + \beta + \sum_{i=1}^{n} \frac{1}{x_i})}{(\beta + \sum_{i=1}^{n} \frac{1}{x_i})(\alpha n + 2 + \beta + \sum_{i=1}^{n} \frac{1}{x_i})},
\]

(38)

From

\[
E[\lambda | x_1, ..., x_n] = \bar{\lambda}.
\]

(39)

Then

\[
E[x_1, ..., x_n | \lambda] = \frac{\lambda}{\alpha - 1}.
\]

(40)

Therefore

\[
E[x_1, ..., x_n | \lambda] = \frac{(\alpha n + 1)(\alpha n + 2 + \gamma + \sum_{i=1}^{n} \frac{1}{x_i})}{(\alpha - 1)(\beta + \sum_{i=1}^{n} \frac{1}{x_i})(\alpha n + 2 + \beta + \sum_{i=1}^{n} \frac{1}{x_i})}.
\]

(41)

The final premium based on the Poisson-Akash distribution for the frequency of the claim and the Inverse-Gamma Lindley for the claim’s severity that the policyholder must pay is

\[
\text{Premium}_{t+1} = \frac{(n+1)(n+2)(n+3)+n+1)(t+\gamma)^2}{(t+\gamma)^3 + (t+\gamma)(n+2)(n+1)}
\]

(42)

\[
\ast \frac{(\alpha n + 1)(\alpha n + 2 + \beta + \sum_{i=1}^{n} \frac{1}{x_i})}{(\alpha - 1)(\beta + \sum_{i=1}^{n} \frac{1}{x_i})(\alpha n + 2 + \beta + \sum_{i=1}^{n} \frac{1}{x_i})}
\]

Now, by applying the Poisson-Akash distribution under the Linex loss function for the claim frequency and the Inverse-Gamma Lindley distribution for the claim severity, the final premium that the policyholder must pay is
$Premium_{t+1} = -\frac{1}{a} \ln \left( \frac{(n+2)(n+1)+(t+\gamma+a)}{(t+\gamma+a)^{n+3}} \right) \cdot \frac{(t+\gamma)^{n+3}}{(n+2)(n+1)+(t+\gamma)^2}$

$\cdot \frac{(an+1)(an+2+\beta+\sum_{i=1}^{n} \frac{1}{x_i})}{(a-1)(\beta+\sum_{i=1}^{n} \frac{1}{x_i})(an+2+\beta+\sum_{i=1}^{n} \frac{1}{x_i})}$

4 NUMERICAL APPLICATIONS

For the numerical application, we used a real data set to compare the premiums based only on the number of claims with those based on both claims frequency and severity of the claim, we examined the data set based on one-year automobile insurance policies taken out in 2004 or 2005. This data set is accessible on the website of the Faculty of Business and Economics, Macquarie University (Sydney, Australia), see also de Jong and Heller (2008). In total, the portfolio contains 67,856 policies, and 4,624 of these policies make at least one claim. Among these policies, 4,333 made one claim, 271 made two claims, 18 made three claims, and 2 made four claims.

4.1 BONUS-MALUS PREMIUMS BASED ON THE CLAIM FREQUENCY COMPONENT ONLY

For the first table, we use the Poisson-Akash distribution introduced earlier in this paper to calculate the claim frequency distribution. $\gamma=14.0125$ is the maximum likelihood estimator of the parameter of the Poisson-Akash distribution in (4). The Bayesian bonus-Malus premiums based only on the claim frequency are calculated from (22), and the results are shown in the following table.
Table 1. Computation of the premiums for Bonus-Malus system using the Poisson-Akash distribution.

<table>
<thead>
<tr>
<th>t</th>
<th>Number of claims</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>93.10335</td>
</tr>
<tr>
<td>2</td>
<td>87.10775</td>
</tr>
<tr>
<td>3</td>
<td>81.84590</td>
</tr>
<tr>
<td>4</td>
<td>77.18976</td>
</tr>
<tr>
<td>5</td>
<td>73.03964</td>
</tr>
<tr>
<td>6</td>
<td>69.31671</td>
</tr>
<tr>
<td>7</td>
<td>65.95779</td>
</tr>
</tbody>
</table>

Source: Authors

The premiums in the Bayesian bonus-malus system increase in the event of a claim and decrease when the insured has a claim-free year. According to the findings in Table 1, a policyholder who file done claim in the first year will be required to pay a malus equal to 87.73% of the base premium. A bonus of 6.89% of the base premium will be added to the premium for the first year with no claims.

To calculate the Bayesian bonus-malus premiums based on claim frequency only, we use the Poisson-Akash distribution under the Linex loss function in (22). We took $\alpha=1.1$ (over estimation). The following table displays the premium results.

Table 2. Computation of the Bonus-Malus Premiums using the Poisson-Akash distribution under Linex Loss Function.

<table>
<thead>
<tr>
<th>t</th>
<th>Number of claims</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>96.13338</td>
</tr>
<tr>
<td>1</td>
<td>89.74638</td>
</tr>
<tr>
<td>2</td>
<td>84.16501</td>
</tr>
<tr>
<td>3</td>
<td>79.24454</td>
</tr>
<tr>
<td>4</td>
<td>74.87319</td>
</tr>
<tr>
<td>5</td>
<td>70.96317</td>
</tr>
<tr>
<td>6</td>
<td>67.44461</td>
</tr>
<tr>
<td>7</td>
<td>64.26111</td>
</tr>
</tbody>
</table>

Source: Authors

As we can see, the first premium at $t = 0$ is 96.13338. A claim made by a policyholder in the first year will result in a malus payment of 80.738718 % of the base premium. A bonus of 6.387% of the basic premium will be added to the premium for the first year with no claims.

The Bayesian bonus-malus premiums based solely on claim frequency are calculated using the Poisson-Akash distribution under the Linex loss function in (22). We took $\alpha$ as -0.3 (under estimation). The premium results are presented in the table below.
Table 3. Computation of the Bonus-Malus Premiums using the Poisson-Akash distribution under Linex Loss Function.

<table>
<thead>
<tr>
<th>t</th>
<th>Number of claims</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>101.1294</td>
</tr>
<tr>
<td>1</td>
<td>94.07916</td>
</tr>
<tr>
<td>2</td>
<td>82.59612</td>
</tr>
<tr>
<td>3</td>
<td>77.85570</td>
</tr>
<tr>
<td>4</td>
<td>73.63484</td>
</tr>
<tr>
<td>5</td>
<td>69.85195</td>
</tr>
<tr>
<td>6</td>
<td>66.44175</td>
</tr>
</tbody>
</table>

Source: Authors

The results in Table 3 show that the initial premium is 101.1294. An insured will have 88.6 % as a malus to pay of the base premium if he makes one claim during the first year. However, having no claim during the first year will be shown in the premium as a bonus at 7.05024 % of the initial premium.

Table 2 presents an over estimation under the Linex loss function for $\alpha=1.1$; the under estimation under the Linex loss function for $\alpha=-0.3$ is shown in Table 3. Comparing the premiums shown in both tables, we notice that the premiums in the overestimation are softer when there is no claim ($n=0$). On the other side, when a policyholder makes claims, we remark that the premiums in the under estimation are stricter with bad drivers compared to the over estimation. The results show that premiums rise rapidly when the number of claims increases. These premiums decrease slowly when there is no claim for the next years.

For comparison purposes, we include the Bayesian bonus-malus premiums calculated using the Poisson-Lindley distribution, as displayed in Table 2 in Moumeesri (2020).

Table 4. Computation of Bonus-Malus premiums using Poisson-Lindley distribution.

<table>
<thead>
<tr>
<th>t</th>
<th>Number of claims</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>93.26</td>
</tr>
<tr>
<td>2</td>
<td>87.37</td>
</tr>
<tr>
<td>3</td>
<td>82.17</td>
</tr>
<tr>
<td>4</td>
<td>77.56</td>
</tr>
<tr>
<td>5</td>
<td>73.43</td>
</tr>
<tr>
<td>6</td>
<td>69.72</td>
</tr>
</tbody>
</table>

Source: Moumeesri (2020)
From the results in Table 4, we remark that an insured who made one claim in the first year will have 85.92% as a malus to pay of the base premium. However, having no claim in the first year will be shown in the premium as a bonus at 6.74% of the base premium.

Comparing Tables 1 to 4, we notice that the premiums using the Poisson-Akash distribution under the Linex loss function in the case of the over estimation ($\alpha = 1.1$) for the frequency only are the softest with good drivers. On the other side, the premiums using the same distribution in case of under estimation ($\alpha = -0.3$) are the most severe for bad drivers.

Moreover, the premiums based on the Poisson-Akash distribution under Linex loss function in the case of under estimation increase rapidly when the number of claims rises. The premiums under this same model decrease slowly with time when there is no claim.

**Remark 1** We noticed that when the value of $\alpha$ tends to 0, the premiums based on the Poisson-Akash distribution under the Linex loss function (over estimation and under estimation) for the claim frequency component will be the exact same premiums as in Table 1 for the premiums based on the Poisson-Akash distribution.

### 4.2 BONUS-MALUS PREMIUMS BASED ON THE FREQUENCY COMPONENT AND LOGNORMAL-GAMMA DISTRIBUTION FOR THE SEVERITY COMPONENT

In this section, we will calculate the bonus-malus premiums based on the Lognormal-Gamma distribution for these verity component and three different distributions for the claim frequency component.

The next table represents the number of claims corresponding to the size of the claims and the total size of the claims. All subsequent tables in this paper will utilize this claim size information.
Table 5. The size of the claim and the total size of claims corresponding to the number of claims.

<table>
<thead>
<tr>
<th>Number of claims</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of the claim</td>
<td>235</td>
<td>471</td>
<td>706</td>
<td>942</td>
<td></td>
</tr>
<tr>
<td>Total size of the claims</td>
<td>235</td>
<td>706</td>
<td>1412</td>
<td>2354</td>
<td></td>
</tr>
</tbody>
</table>

Source: Moumeesri (2020)

According to Table 5, we can see that the size of one claim will be 235, and the size of two claims will be the accumulation of the first claim (235) and the second claim (471), which is a total of 706, and soon.

The table below displays the bonus-malus premiums, which are calculated using the Poisson-Akash distribution for claim frequency and the Lognormal-Gamma distribution for claim severity.

Table 6. Computation of the Bonus-Malus Premiums using the Poisson-Akash and Lognormal-Gamma distributions.

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>592.9014</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>552.0111</td>
<td>1052.479</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>516.4631</td>
<td>1418.287</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>485.2655</td>
<td>1330.686</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>457.6592</td>
<td>1253.428</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>433.0531</td>
<td>1184.767</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>410.9797</td>
<td>1123.333</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>391.0647</td>
<td>1068.031</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source: Authors

Table 6 show that the initial premium is 592.9014, if a policyholder has a claim-free year, the premium will be 552.0111 for the next year. But if this policyholder has a claim during the first year, the premium that must be paid for the next year will be 1052.479.

We also note that premiums increase quickly as the number of claims increases. For example, the premium for making one claim during the year is 1052.479, but for making four claims during the same year, the premium will be 2396.876. On the other hand, the premiums gradually decrease over time when there is no claim.

The next table shows the bonus-malus premiums using the Poisson-Akash distribution under the Linex loss function for the claim frequency and the Lognormal-Gamma distribution for the claim severity. We took \( \alpha=1.1 \) (over estimation).
The results show that the initial premium for an insured is 569.9762; for a policyholder who had a claim-free year, the premium will be 532.1075 for the following year. However, if the policyholder makes a claim in the first year, they will have to pay a premium of 1014.015 for the next year. We notice that when there is no claim, the premiums decrease slowly with time. However, as the number of claims increases, the premiums rise. As illustrated, if a policyholder makes a claim, he will pay 1043.671, but if he makes four claims, he will pay 2306.882. When there is no claim for the next years, the premium at $t = 7$ will be 1680.376.

The next table shows the bonus-malus premiums using the Poisson-Akash distribution under the Linex loss function for the claim frequency and the Lognormal-Gamma distribution for the claim severity. We took $\alpha = -0.3$ (underestimation).

Table 8. Computation of the Bonus-Malus Premiums using the Poisson-Akash distribution and Lognormal-Gamma distributions.

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>599.5978</td>
<td>1063.673</td>
<td>1534.831</td>
<td>1984.794</td>
<td>2423.105</td>
</tr>
<tr>
<td>1</td>
<td>557.7966</td>
<td>993.5497</td>
<td>1432.499</td>
<td>1851.275</td>
<td>2259.030</td>
</tr>
<tr>
<td>2</td>
<td>489.7136</td>
<td>932.2276</td>
<td>1343.158</td>
<td>1734.832</td>
<td>2116.011</td>
</tr>
<tr>
<td>3</td>
<td>461.6075</td>
<td>878.1322</td>
<td>1264.464</td>
<td>1632.369</td>
<td>1990.236</td>
</tr>
<tr>
<td>4</td>
<td>436.5820</td>
<td>830.0456</td>
<td>1194.603</td>
<td>1541.496</td>
<td>1878.755</td>
</tr>
<tr>
<td>5</td>
<td>414.1532</td>
<td>787.0109</td>
<td>1132.156</td>
<td>1460.339</td>
<td>1779.253</td>
</tr>
<tr>
<td>6</td>
<td>393.9341</td>
<td>748.2649</td>
<td>1075.992</td>
<td>1387.408</td>
<td>1689.888</td>
</tr>
</tbody>
</table>

Source: Authors

According to Table 8, the initial premium is 599.5978. If a policyholder makes a claim, the premium for the next year will be 1063.673. This premium will keep increasing until it reaches 2423.105 for making four claims. If a policyholder makes no claim any longer, the premium will be 1689.888 at time $t = 7$. The results
show that the premiums increase quickly when the number of claims increases. These premiums decrease slowly when there is no claim for the next years.

Table 7 presents an over estimation under the Linex loss function for $\alpha = 1.1$, and Table 8 shows an under estimation under Linex loss function for $\alpha = -0.3$. Comparing the results in both tables, we observe that the premiums in the under estimation are higher when there is no claim ($n = 0$). Conversely, when an insured makes claims, we notice that the under estimation of premiums is more severe than the over estimation. For comparison, we introduce in the next table, the Bayesian bonus-malus premiums based on the Poisson-Lindley distribution for the frequency component and Lognormal-Gamma distribution for the severity component, as shown in Table 2 in Moumeesri (2020).


<table>
<thead>
<tr>
<th>t</th>
<th>Number of claims</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>592.53</td>
</tr>
<tr>
<td>1</td>
<td>552.60</td>
</tr>
<tr>
<td>2</td>
<td>517.67</td>
</tr>
<tr>
<td>3</td>
<td>486.90</td>
</tr>
<tr>
<td>4</td>
<td>459.56</td>
</tr>
<tr>
<td>5</td>
<td>435.12</td>
</tr>
<tr>
<td>6</td>
<td>413.14</td>
</tr>
<tr>
<td>7</td>
<td>393.26</td>
</tr>
</tbody>
</table>

Source: Authors

The initial premium for an insured in Table 9 is 592.53. A policyholder who didn’t have a claim during the year will pay 552.60 for the next year. However, the premium for the next year will be 1041.67 if this insured makes a claim.

We remark that the premiums increase slowly when the number of claims increases. On the other hand, when there is no claim, the premiums decrease rapidly. From tables 6 to 9, we observe that the model based on the Poisson-Akash distribution under the Linex loss function in case of an over estimation ($\alpha = 1.1$) for the frequency component and the Lognormal-Gamma for the claim severity is the most generous with good drivers. However, the premiums utilizing the same model in the case of the under estimation ($\alpha = -0.3$) are stricter for bad drivers.

Furthermore, the premiums based on the Poisson-Akash distribution under the Linex loss function in the case of under estimation rise quickly when the number of claims rises; these premiums decrease rapidly with time when there is
no claim. We also notice that when there is no claim, the model based on the Poisson-Lindley distribution decreases slowly with time.

In addition, we remark that the premiums using the Poisson-Akash distribution under the Linex loss function in case of an over estimation and the premiums using the Poisson-Lindley are slightly different for bad drivers, especially when the policyholder makes four claims. However, we observe that the Poisson-Lindley premiums decrease slowly when there is no claim.

4.3 BONUS-MALUS PREMIUMS BASED ON THE CLAIM FREQUENCY COMPONENT AND INVERSE-GAMMA LINDLEY DISTRIBUTION FOR THE SEVERITY COMPONENT

In this section, we present the bonus-malus premiums using the inverse-Gamma Lindley distribution presented earlier in this paper for the claim severity with different claim frequency distributions. The parameters of the Inverse-Gamma Lindley distribution in (26) are: $\hat{\alpha} = 1.08$ and $\hat{\beta} = 0.001766$.

The following table shows the bonus-malus premiums based on the Poisson-Akash distribution for the claim frequency and the Inverse-Gamma Lindley distribution for the claim severity using the formula (42).

<table>
<thead>
<tr>
<th>t</th>
<th>Number of claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>515.3173</td>
</tr>
<tr>
<td>1</td>
<td>479.7776</td>
</tr>
<tr>
<td>2</td>
<td>448.8813</td>
</tr>
<tr>
<td>3</td>
<td>421.7660</td>
</tr>
<tr>
<td>4</td>
<td>397.7722</td>
</tr>
<tr>
<td>5</td>
<td>376.3859</td>
</tr>
<tr>
<td>6</td>
<td>357.2010</td>
</tr>
<tr>
<td>7</td>
<td>339.8919</td>
</tr>
</tbody>
</table>

The results indicate that the initial premium for the Poisson-Akash distribution is 515.3173. This premium will decrease to 479.7776 if a policyholder has a claim-free year. The premium will increase to 590.1701 if this policyholder claims during the first year.
This premium will keep rising quickly with the increase in the number of claims to reach 2183.010 after four claims. If the policyholder does not claim for the next years, the premium will decrease slowly to attain 1527.576.

The table below describes the bonus-malus premiums based on the Poisson-Akash distribution under the Linex loss function for the claim frequency and the Inverse-Gamma Lindley distribution for the claim severity by applying the formula in 43. We took $\alpha=1.1$ (over estimation).

<table>
<thead>
<tr>
<th>t</th>
<th>Number of claims</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>495.3919</td>
</tr>
<tr>
<td>1</td>
<td>433.7168</td>
</tr>
<tr>
<td>2</td>
<td>408.3608</td>
</tr>
<tr>
<td>3</td>
<td>385.8345</td>
</tr>
<tr>
<td>4</td>
<td>365.6855</td>
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<tr>
<td>5</td>
<td>347.5537</td>
</tr>
<tr>
<td>6</td>
<td>331.1486</td>
</tr>
</tbody>
</table>

Source: Authors

The base premium shown in Table 11 of the Poisson-Akash distribution under the Linex loss function is 495.3919; the premium rises to 568.6013 when the insured claims during the year. After a year without a claim, the premium for the following year will be 462.4786. According to the results in Table 11, we observe that when the number of claims augments, the premiums swiftly rise to attain 2101.046 if the insured makes four claims. These premiums will decrease slowly with time.

The table below describes the bonus-malus premiums based on the Poisson-Akash distribution under the Linex loss function for the claim frequency and the Inverse-Gamma Lindley distribution for the claim severity by applying the formula in 43. We took $\alpha=-0.3$ (under estimation).
Table 12. Computation of the Bonus-Malus Premiums using the Poisson-Akash and Inverse-Gamma Lindley distributions.

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>521.1374</td>
<td>596.4468</td>
<td>1012.625</td>
<td>1553.915</td>
<td>2206.899</td>
</tr>
<tr>
<td>1</td>
<td>484.8061</td>
<td>557.1258</td>
<td>945.1103</td>
<td>1449.381</td>
<td>2057.463</td>
</tr>
<tr>
<td>2</td>
<td>453.2708</td>
<td>522.7398</td>
<td>886.1668</td>
<td>1358.216</td>
<td>1927.206</td>
</tr>
<tr>
<td>3</td>
<td>425.6321</td>
<td>492.4062</td>
<td>834.2470</td>
<td>1277.997</td>
<td>1812.653</td>
</tr>
<tr>
<td>4</td>
<td>401.2038</td>
<td>465.4420</td>
<td>788.1557</td>
<td>1206.852</td>
<td>1711.119</td>
</tr>
<tr>
<td>5</td>
<td>379.4530</td>
<td>441.3106</td>
<td>746.9554</td>
<td>1143.314</td>
<td>1620.495</td>
</tr>
<tr>
<td>6</td>
<td>359.9591</td>
<td>419.5841</td>
<td>709.9004</td>
<td>1086.215</td>
<td>1539.104</td>
</tr>
<tr>
<td>7</td>
<td>342.3858</td>
<td>419.5841</td>
<td>709.9004</td>
<td>1086.215</td>
<td>1539.104</td>
</tr>
</tbody>
</table>

Source: Authors

Table 12 shows that the base premium is 521.1374. The premium will rise to 596.4468 if the insured makes a claim; this premium will keep rising to attain 2206.899 when the insured makes four claims. If this insured does not claim for the following years, the premium at time $t = 7$ will be 1539.104.

From the results in Table 12, we notice that the premiums augment rapidly with the rise in the number of claims. These premiums decrease slowly when there is no claim. The Table 11 shows an over estimation with $\alpha=1.1$, and the Table 12 presents an under estimation with $\alpha = – 0.3$. We observe that in the over estimation, the premiums when there is no claim ($n = 0$) are softer compared to the under estimation. However, the premiums in the under estimation when the policyholder makes claims are stricter and larger compared to the over estimation. The next table represents the bonus-malus premiums based on the Poisson-Lindley distribution from Moumeesri (2020) for the frequency component and the Inverse-Gamma Lindley distribution for the severity component.

Table 13. Computation of the Bonus-Malus Premiums using the Poisson-Lindley and Inverse-Gamma Lindley distributions.

<table>
<thead>
<tr>
<th>t</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>514.9946</td>
<td>584.1088</td>
<td>981.2601</td>
<td>1491.545</td>
<td>2100.599</td>
</tr>
<tr>
<td>1</td>
<td>480.2682</td>
<td>547.3892</td>
<td>919.8378</td>
<td>1398.526</td>
<td>1970.016</td>
</tr>
<tr>
<td>2</td>
<td>449.9433</td>
<td>514.9887</td>
<td>865.6050</td>
<td>1316.350</td>
<td>1854.603</td>
</tr>
<tr>
<td>3</td>
<td>423.1892</td>
<td>486.1908</td>
<td>817.3748</td>
<td>1243.235</td>
<td>1751.876</td>
</tr>
<tr>
<td>4</td>
<td>399.4262</td>
<td>460.4284</td>
<td>774.2070</td>
<td>1177.768</td>
<td>1659.861</td>
</tr>
<tr>
<td>5</td>
<td>378.1808</td>
<td>437.2474</td>
<td>735.3475</td>
<td>1118.812</td>
<td>1576.973</td>
</tr>
<tr>
<td>6</td>
<td>359.0742</td>
<td>416.2795</td>
<td>700.1844</td>
<td>1065.447</td>
<td>1501.923</td>
</tr>
</tbody>
</table>

Source: Authors

The results illustrated in the above table show that the premium at $t=0$ for the Poisson-Lindley distribution is 514.9946. If the policyholder does not claim
during the first year, the premium will be reduced to 480.2882. If not, the premium will increase to 584.1088.

These premiums will keep increasing slowly with the rise in the number of claims reaching 2100.599 for four claims made during the year. In case the policyholder does not claim during the next years, the premium will decrease rapidly, reaching 1501.923.

According to Tables [10-13], we remark that the premiums using the Poisson-Akash distribution under the Linex loss function in the case of an overestimation ($\alpha=1.1$) for the claim frequency, and the Inverse-Gamma Lindley distribution for the claim severity are the softest for bad drivers. However, the premiums using the same model in case of an underestimation ($\alpha=-0.3$) are more severe for bad drivers.

Moreover, we observe that the premiums based on the Poisson-Akash distribution under the Linex loss function in the case of underestimation increase rapidly when the number of claims rises and decrease quickly when there is no claim. On the other hand, we remark that the premiums based on the Poisson-Lindley model decrease slowly when there is no claim.

5 CONCLUSION

This study introduces a novel approach for determining bonus malus premiums by employing the Bayesian method. This model incorporates both the frequency and severity of claims. We employed a combination of the Poisson distribution with the Akash distributions for the claim frequency component and a new combination of the Inverse-Gamma distribution with Lindley distributions for the claim severity component. The premiums were calculated using the Bayesian technique. In this study, we additionally presented the Linex loss function and computed the bonus-malus premiums using this loss function for both claim frequency and claim severity.

We employed a numerical application utilizing an actual data set of automotive insurance to demonstrate our approach. Based only on claim frequency, the results indicate that the Poisson-Akash distribution, when evaluated using the Linex loss function, provides more versatility compared to alternative
distributions. The analysis of premiums, taking into account claim frequency and claim severity, indicates that the Poisson-Akash distribution, when evaluated using the Linex loss function, offers more advantages for drivers with good records. Nevertheless, we noticed a rapid escalation in premiums when the number of claims increased, specifically when using the Poisson-Akash distribution for frequency and the Inverse-Gamma Lindley distribution for claim severity. This model exhibits a gradual reduction in value in the absence of any claims in the subsequent years. We observed that the model utilizing the Poisson-Akash distribution for frequency and the Lognormal-Gamma distribution for claim severity is more stringent when dealing with bad drivers, in contrast to the model employing the same distribution for claim frequency and the Inverse-Gamma Lindley distribution for claim severity, which is more lenient with good drivers. These models provide appropriate incentives for excellent drivers and impose suitable penalties on bad drivers.

Furthermore, it is worth mentioning that the premiums in the underestimate for the model using the Poisson-Akash distribution under the Linex loss function for claim frequency and the Inverse-Gamma Lindley distribution for claim severity are higher and more stringent when it comes to problematic drivers. Conversely, the premiums for the underestimate in the bonus-malus model, which is based on the Poisson-Akash distribution for claim frequency and the Lognormal-Gamma distribution for claim severity, are greater. Additionally, the premiums for bad drivers are more substantial. This paper presented the establishment of fair and equitable premiums through the use of the bonus malus system. The system relies on the Poisson Akash distribution to determine claim frequency and the Inverse Gamma Lindley distribution to assess claim severity. This adjusts premiums to adequately cover policyholders’ risk and ensures the insurance company's solvency remains intact. Our research also shows that by adjusting the value of $\alpha$ in the Linex loss function, the insurer may adapt rates to the needs of the business. We aim to calculate the bonus-malus premiums using new distributions and alternative loss functions, such as the entropy loss function, in order to avoid the use of explicit estimators and the need for simulations.
REFERENCES


