American option in a market generated by interactive agents

Opção americana em um mercado gerado por agentes interativos

DOI: 10.54021/seesv5n1-163

Recebimento dos originais: 21/05/2024
Aceitação para publicação: 11/06/2024

Besma Hamidane
PhD Students in Mathematics
Institution: LaPS Laboratory, Badji Mokhtar Annaba-University
Address: P.O. Box 12, Annaba 23000, Algeria
E-mail: besma.hamidane@univ-annaba.dz

Nawel Arrar
PhD in Mathematics
Institution: National School of Artificial Intelligence; LaPS Laboratory, Badji Mokhtar Annaba-University
Address: Sidi Abdallah, Algiers, Algeria
E-mail: nawel.remita@ensia.edu.dz

Mohamed Riad Remita
PhD in Mathematics
Institution: National School of Artificial Intelligence; LaPS Laboratory, Badji Mokhtar Annaba-University
Address: Sidi Abdallah, Algiers, Algeria
E-mail: riad.remita@ensia.edu.dz

ABSTRACT

This study introduces a new financial market model inspired by Remita and Eisele's methodology, incorporating a substantial number of interacting agents denoted by “n”. Focusing on the valuation of European and American options within this vast network of agents, and more specifically on behavior when “n” tends to infinity, the research illustrates the convergence of an initially incomplete market towards completeness. In addition, the study scrutinizes the price stability of options on the underlying risky asset, St, in the context of an increasing number of agents. Taking into account a comprehensive set of internal and external factors influencing market dynamics, this study offers a holistic analysis of option price dynamics and market completeness. By examining the complex interactions between a large number of agents and option pricing, the research provides valuable insights into the complexity of financial markets. The results not only elucidate convergence phenomena within an expanding network of agents, they also shed light on the stability of option prices under ever-changing market conditions. This comprehensive analysis underlines the multifaceted nature of financial markets, highlighting the intrinsic relationship between option price dynamics and various market influences. Overall, this study makes a significant
contribution to the understanding of financial market dynamics, particularly in the context of option pricing.

**Keywords:** complete market, incomplete market, european option, american option, risk neutral probability.

**RESUMO**

Este estudo apresenta um novo modelo de mercado financeiro inspirado na metodologia de Remita e Eisele, incorporando um número substancial de agentes que interagem, denotados por “n”. Concentrando-se na avaliação de opções europeias e americanas dentro dessa vasta rede de agentes e, mais especificamente, no comportamento quando “n” tende ao infinito, a pesquisa ilustra a convergência de um mercado inicialmente incompleto em direção à completude. Além disso, o estudo examina a estabilidade do preço das opções sobre o ativo de risco subjacente, St, no contexto de um número crescente de agentes. Levando em conta um conjunto abrangente de fatores internos e externos que influenciam a dinâmica do mercado, este estudo oferece uma análise holística da dinâmica do preço das opções e da integridade do mercado. Ao examinar as interações complexas entre um grande número de agentes e a precificação de opções, a pesquisa fornece informações valiosas sobre a complexidade dos mercados financeiros. Os resultados não só elucidan os fenômenos de convergência em uma rede de agentes em expansão, como também esclarecem a estabilidade dos preços das opções em condições de mercado em constante mudança. Essa análise abrangente ressalta a natureza multifacetada dos mercados financeiros, destacando a relação intrínseca entre a dinâmica dos preços das opções e as diversas influências do mercado. De modo geral, esse estudo contribui significativamente para a compreensão da dinâmica do mercado financeiro, especialmente no contexto da precificação de opções.

**Palavras-chave:** mercado completo, mercado incompleto, opção europeia, opção americana, probabilidade neutra em termos de risco.

**1 INTRODUCTION**

Mathematical finance is a fast-growing field of study that underpins not only current, but also modern financial practice. The pioneering work of Louis Bachelier [1] on the Theory of Speculation in 1900 and that of Samuelson [12](1965) on geometric Brownian motion laid the foundations of this discipline. The development of derivatives markets began with the work of Black and Scholes [2] (1973), establishing the standard option pricing model known as the Black Scholes Model (BSM).

However, the Gaussian assumption for the return on financial assets has well-known limitations. Alternative models were proposed as early as the 1960s by Mandelbrot [9] (1963) and Fama [6] (1965), and later by Merton [10] (1976) with his non-Gaussian model. Since the 1990s, a profusion of alternative models has
emerged, including Lévy's exponential processes such as Normal Inverse Gaussian, Meixner, Generalized Hyperbolic, Gamma Variance and CGMY. Reference works such as those by Cont and Tankov [3] (2004) or Eberlein [4] (2007) have contributed to this diversification.

In a modeling framework using processes with discontinuous trajectories, the challenge of market incompleteness arises, as explored by researchers such as Fujiwara and Miyahara [7] (2003), as well as Cont and Tankov [3] (2004). Merton, in his model, introduced a mixed process with a Gaussian component and a Poisson component made up of Gaussian jumps, while Kou [8] (2002) proposed a similar approach with jumps following an asymmetric double exponential distribution, thus enabling the evaluation of exotic options.

Another innovative approach was presented by Remita and Eisele [11] (2006), who developed a model taking into account the interaction of agents in price formation, inspired by the work of Eisele and Ellis [5] (1988) on phase transitions in the generalized Curie-Weiss model. By considering the interactions between agents in the financial market, this model offers an interesting perspective on the transition from an incomplete to a complete market.

In this article, we propose to use Remita and Eisele’s model to value American options, an aspect less explored in the literature, highlighting the convergence towards a complete market as the number of agents increases. We begin by briefly introducing Remita and Eisele's model, then outline risk-neutral probabilities for valuing American options under the "n" agent model and in the limit case. Finally, we conclude by examining the convergence of risk-neutral probabilities and option pricing as the number of agents tends to infinity.

2 MODEL CONSTRUCTION

We begin first by describing briefly the model proposed by Remita and Eisele, where they supposed that the market is composed from agents noted “i” with i = 1,…,n; (n going eventually to infinity). Each agent has his personal opinion about the future evolution of the stock price on the market. The opinion is built individually from informations about the enterprise, whose the stock is the indicator, it depends also on external events, general circumstances, economic indicators, inflation, budget deficit, money supply, raw materials' prices and
behavior of other stock markets. This opinion is expressed by an individual drift depending on time, which will be noted by $\mu_i(t)$. This means that the agent $i$ perceive the future evolution of the stock price given by

$$S_{i,t} = S_0 \mathcal{E}(X_t) \exp\{\mu_i(t) \cdot t\}$$

With

$$\mathcal{E}(X_t) = \exp\left\{X_t - \frac{1}{2} [X,X]_t \right\} \prod_{0 < s \leq t} (1 + \Delta X_s) \exp\{-\Delta X_s\},$$

The stochastic exponential of the semimartingale $X_t$, where $[X,X]_t$ is the continuous part of $[X,X]$, and we suppose that $\Delta X_s > -1$ for all $t \in [0, T]$. $S_0$ is the actual price of the stock, $\mu_i(t)$ the subjective opinion of the agent “i” on the future evolution of the stock price, and $X_t$ the external stochastic perturbation (the circumstances). Then, $X_t$ represents all external stock market data that have direct influence on prices, it will be for example economists’ and politicians’ opinions, we suppose $X_0 = 0$. Then the sequence of price processes $(S_i(\mu_i))_{i=1, \ldots, n}$ is a family of processes in a random environment. At time $t$ the price $S^n_t$ of the stock is the result of the estimations of the $n$ agents; more precisely, it should be the median of the agents' estimations $S_{i,t}$ of agents $i$, weighted by the quantities of stock offered or required by the agent $i$. Because we suppose that the agent $i$ buys stocks if the official price $S^n_t$ on the market is less than the individual price $S_{i,t}$ and he sells if $S^n_t \geq S_{i,t}$. Then the market is in equilibrium if $S^n_t$ is the weighted median of the $S_{i,t}$. Then

$$\psi(S^n_t) = \frac{1}{n} \sum_{i=1}^{n} \psi(S_{i,t}).$$  \hspace{1cm} (1)

In order to make the proofs easy, we will study the case where $\psi(x) = x$ and will give in the following the general result.

To explain the dynamic of the market, we consider a stock whose subjective price is estimated by an agent “i”. Under the influence of the opinions of the other agents and external news, the agent will change his opinion from time to time; this is done by a change of his individual drift $\mu_i(t)$. $S_0$, conditionally to $X_{[0,T]}$, the pure
jump process \((\mu(n)(t))_{0 \leq t \leq T} = \mu^{(n)}(t) = (\mu_1(t), \ldots, \mu_n(t)) \in B^n\), which represent the configuration of the opinions, has the conditional infinitesimal generator \(A = A(X_t)\) defined on product functions \(f(b_1, \ldots, b_n) = \prod_{i=1}^{n} f(b_i), f \in C_b(B)\), by

\[
(Af) \left( \mu^{(n)}(t) \right)
= \sum_{i=1}^{n} \left( \prod_{j \neq i} f_j(\mu_j(t)) \times \int_B [f_i(m) - f_i(\mu_i(t))] \right)
\times \exp \left\{ \alpha + \frac{1}{n} \sum_{k=1}^{n} g(m, \mu_k(t)) + my X_t \right\} \rho(dm)
\]

with \(\alpha, \gamma\) are positive constants and \(g \in C_b(B^2)\).

The number \(\alpha\) represents the level of the nervousness of the market, (general intensity of the agents to change their mind). The term \(\frac{1}{n} \sum_{k=1}^{n} g(m, \mu_k(t))\) represents the interaction with the other agents. \(my X_t\) is the influence of external events. The probability measure \(\rho\) is the distribution of the new drift.

Then, conditionally to \(X_{[0,T]}\), it exists a unique Markov process \(\mathbb{P}^n(\cdot|X.)\) on the Skorokhod space \(D(\mathbb{R}+, B^n)\), the space of càdlàg processes with values in \(B^n\). Let \(\mathbb{P}^n\) the process defined on the space \(\Omega = D(\mathbb{R}+, B^n \times \mathbb{R})\). Then we have for \(f(b_1, \ldots, b_n) = \prod_{i=1}^{n} f(b_i), f \in C_b(B)\)

\[
M^n_t = f \left( \mu^{(n)}(t) \right) - f \left( \mu^{(n)}(0) \right)
- \int_0^t \sum_{i=1}^{n} \left( \prod_{j \neq i} f_j(\mu_j(s)) \times \int_B [f_i(m) - f_i(\mu_i(s))] \right)
\times \exp \left\{ \alpha + \frac{1}{n} \sum_{k=1}^{n} g(m, \mu_k(s)) + my X_s \right\} \rho(dm) \right) ds
\]

a \(\mathbb{P}^n\)-martingale conditionally to \(X\). We set \(\omega^{(n)}_t = (\mu^{(n)}(t), X_t)\).
Writing $M^n_t = \int_0^t \int_B [f(\bar{m}) - f(\mu^n(s))] \tilde{A}(dm, ds)$, where $\bar{m} \in B^n$

\[ \tilde{A}(dm, ds) = A(dm, ds) \]

\[ - \sum_{i=1}^n \int_B \delta_{(\mu_i(s), \mu_{i-1}(s), \mu_{i+1}(s), \ldots, \mu_n(s))} \times \exp \left\{ \alpha + \frac{1}{n} \sum_{k=1}^n g(m, \mu_k(s)) + m\gamma X_s \right\} \rho(dm) ds. \]

$A(dm, ds)$ is, conditionally to $X_t$, a pure point process, the corresponding increasing process of $M^n_t$ will be given by

\[ \langle M^n, M^n \rangle_t = \int_0^t \int_B [f(\bar{m}) - f(\mu^n(s))]^2 \times \left( \sum_{i=1}^n \int_B \delta_{(\mu_i(s), \mu_{i-1}(s), \mu_{i+1}(s), \ldots, \mu_n(s))} \times C^m(m, X_s) \rho(dm) \right) d(\bar{m}) ds, \]

where $C^m(m, X_s) = \exp \left\{ \alpha + \frac{1}{n} \sum_{k=1}^n g(m, \mu_k(s)) + m\gamma X_s \right\}$.

In Remita and Eisele (2006) we have shown that the empirical measure

$\mu^n_t = \frac{1}{n} \sum_{i=1}^n \delta_{\mu_i(t)}$ converges in law to the process $\mu_t$.

**Theorem 1.** The process $(\mu^n_t)_{t \geq 0}$ converges in law to the process $\mu_t$ satisfying, conditionally to $X_t$, the ordinary differential equation

\[ \left( \frac{d}{dt} \right) \mu_t = \exp\{\alpha + m(\beta m_t + \gamma X_t)\} - \mu_t \exp\{\alpha + \varphi(\beta m_t + \gamma X_t)\}, \]

where $m_t = \int_B m \mu_t(dm)$ is the mean of the opinions and $\varphi$ is the logarithm of the moment generating function.

From Theorem 1. we deduce then the following result.

**Corollary 1.** The price process $(S^n_t)_{t \geq 0}$ converges in law to the process $(S_t)_{t \geq 0}$ defined by $S_t = S_0 \cdot \mathbb{E} \int_B \exp \{mt\} \mu_t(dm)$. 
If we consider the particular case where $X_t$ is a diffusion process, we get the price process given by P. A. Samuelson. The main remark we can do here is that the market, who was incomplete, converges to a complete one.

3 HEDGING EUROPEAN OPTIONS FOR GREAT NUMBER OF AGENTS

3.1 INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P}^n)$ the reference probability space. We recall that $\Omega = D([0, T], \mathbb{P}(B) \times \mathbb{R})$ where $\mathbb{P}(B)$ is endowed with the weak-* topology. $\mathbb{P}^n$ is the objective probability (or historic) for which the process with values in $\mathbb{P}(B)$

$$M^n_t = \mu^n_t - \mu^n_0 - \int_0^t F(\mu^n_s)ds$$

with

$$F(\mu^n_s) = \int_{B^2} (\delta_m - \delta_m) C(m, \mu^n_s) \mu^n_s(d\bar{m}) \rho(dm) ds$$

and

$$C(m, \mu^n_s) = \exp\{\alpha + \beta(g(m, .), \mu^n_s) + \gamma X_s\}$$

is such that $(f, M^n_t)$ is a martingale for all function $f \in C(B)$. We suppose that the filtration $\mathcal{F}$ is minimal with respect to processes $\mu^n_t$ and $X_t$, and realize the used conditions. We will consider call options and suppose two assets are treated in the market $S^2$ and $S^n$ on the period $[0, T]$ where $T$ is the maturity of the option. $S^2$ is the riskless asset (cash) which the value at date $t$ is defined by $S^2_t = e^{rt}$ where $r$ is the constant interest rate and, $S^n$ is the risky asset (action) which price at date $t$ is defined by

$$S^n_t = S_0 \mathcal{E}_t X. \int_B \exp\{mt\} \mu^n_t(dm).$$

We suppose that assets don’t pay dividends, there is no transaction costs and we can sell short without restriction.
3.2 RISK NEUTRAL PROBABILITIES

Let \( \tilde{S}_t^n \) the discounted price of the risky asset, with the constant interest rate \( r > 0 \), which is defined by
\[
\tilde{S}_t^n = e^{-rt} S_t^n,
\]
then \( \tilde{S}_t^n = S_0 \mathcal{E}_t X (f_t(\bar{m}), d\mu_\mathcal{F}(\bar{m})) \), where \( f_t(\bar{m}) = \exp\{\bar{m} - rt\} \) and \( (f_t(\bar{m}), d\mu_\mathcal{F}(\bar{m})) = \int_B f_t(\bar{m}), \mu_\mathcal{F}(d\bar{m}) \).

In the following we take \( X_t \) diffusion process satisfying
\[
dX_t = a(t)dt + b(t)dW_t,
\]
where \( W_t \) is a Wiener process. \( a \) and \( b \) are deterministic and continuous functions on \([0, T]\). We suppose also that
\[
\inf_{t \in [0,T]} b(t) > 0.
\]

Of course we can easily generalize this condition. We take now the "martingale" \( M^n_t \) with values in \( P(B) \) defined in (3) and which will be written using the point process \( \Lambda(d\bar{m}, ds) \).

We get
\[
M^n_t = \int_0^t \int_{B^n} (\mu_m - \mu^n_s) \Lambda(d\bar{m}, ds),
\]
where for \( \bar{m} = (m_1, \ldots, m_n) \) we have put \( \mu_m = \frac{1}{n} \sum_{i=1}^n \delta_{m_i} \), (8) will be simplified if we put
\[
M^n_t = \int_0^t \int_{M(B^n)} \nu \Gamma(n)(d\nu, ds).
\]

The square integrable martingales brackets' \( \langle f, M^n_t \rangle \) and \( \langle g, M^n_t \rangle \) with \( f, g \in C(B) \) are
\[
\langle \langle f, M^n \rangle, \langle g, M^n \rangle \rangle_t = \frac{1}{n} \int_0^t \int_{B^n} \left( f(m) - f(\bar{m}) \right) \left( g(m) - g(\bar{m}) \right) C(m) d\rho(dm) d\mu^n_s(d\bar{m}) ds - \frac{1}{n} \int_0^t \langle \langle f, M^n \rangle, \langle g, M^n \rangle \rangle_t \left( f(m) - f(\bar{m}) \right) \left( g(m) - g(\bar{m}) \right) C(m) d\rho(dm) d\mu^n_s(d\bar{m}) ds.
\]

We write
\[d(\hat{\Gamma}^{(n)}(dv,\cdot),\hat{\Gamma}^{(n)}(dv,\cdot))_s = n \(\langle (\delta_1 - \delta_\bar{m})(dv), \mathcal{C}(m) d\rho(m) d\mu^n_s(\bar{m}) \rangle \) ds. \quad (11)\]

Let two adapted processes \(\beta^n_1(t,v) = \beta^n_1(t,v,\omega)\) and \(\beta^n_2(t,v) = \beta^n_2(t,\omega)\) satisfying the following integrability conditions

\[\int_0^T |\beta^n_2(t)|^2 ds < +\infty \quad \mathbb{P}^n - \text{p.s.} \quad (12)\]

and

\[n \langle (\beta^n_1)^2(s,\frac{1}{n}(\delta_m - \delta_{\bar{m}})), \mathcal{C}(m) d\rho(m) d\mu^n_s(\bar{m}) \rangle ds < +\infty \quad \mathbb{P}^n - \text{p.s.} \quad (13)\]

with

\[\beta^n_1(t,v) > -1. \quad (14)\]

We have then the proposition for the existence of risk neutral probabilities

**Proposition 1.** Risk neutral probabilities \(\mathbb{Q}^n\) (in the system of \(n\) agents) are defined by

\[d\mathbb{Q}^n / \mathcal{F}_t = \mathcal{E}_t N^n. d\mathbb{P}^n / \mathcal{F}_t\]

with

\[N^n_t = \int_0^t \int_{\mathcal{M}(\beta)} \beta^n_1(s,v) \Gamma^{(n)}(dv,ds) + \int_0^t \beta^n_2(s) dW_s, \quad (15)\]

where \(\beta^n_1, \beta^n_2\) realize a.e. the following equation

\[\langle f_t(\bar{m})[a(t) + b(t) \beta^n_2(t) + (\bar{m}) - r], d\mu^n_s(\bar{m}) \rangle + \langle (f_t(m) - f_t(\bar{m})) \left[ 1 + \beta^n_1 \left( t, \frac{1}{n}(\delta_m - \delta_{\bar{m}}) \right) \right], \mathcal{C}(m) d\rho(m) d\mu^n_s(\bar{m}) \rangle \quad (16)\]

The increasing process of \(N^n_t\) is

\[\langle N^n, N^n \rangle_t = \int_0^t (\beta^n_2)^2(s) ds + n \int_0^t \langle (\beta^n_1)^2 \left( s, \frac{1}{n}(\delta_m - \delta_{\bar{m}}) \right), \mathcal{C}(m) d\rho(m) d\mu^n_s(\bar{m}) \rangle ds \quad (17)\]
This model, is generated by two components, macroeconomic one \((X_t)\) and microeconomic one \((\mu^n_t)\). Then it is natural that the change of probability will be on every one. This change will be made using the processes \(\beta^n_1\) and \(\beta^n_2\) in the following way.

a) the process \(\beta^n_1\), acting on the opinions, will be the microeconomic component;

b) the process \(\beta^n_2\), acting on the conjuncture, will be then the macroeconomic component. Since the great number of risk neutral probabilities, the market is then incomplete.

### 3.3 HEDGING EUROPEAN OPTIONS

We will be interested now by hedging options when the price of the underlying asset is defined by \(S^n_t\). We recall that the discounted price process \(\hat{S}^n_t\) is a \(\mathbb{Q}^n\)-martingale if and only if the processes \(\beta^n_1\) and \(\beta^n_2\) satisfy the equation (16). In this case, it follows that with the \(\mathbb{Q}^n\)-martingales

\[
B_t = W_t - \int_0^t \beta^n_2(s)ds
\]  

(18)

and

\[
\int_{\mathcal{M}(B)} (f_s - \nu) \Gamma^n(ds, dv) = \int_{\mathcal{M}(B)} (f_s - \nu) \Gamma^n(ds, dv) \\
- \int_0^t \langle\widetilde{f}_s(m) - f_s(\hat{m}), \mu^n_s(\hat{m})\rangle \beta^n_1(s) \frac{1}{n}(\delta_m - \delta_{\hat{m}}) \cdot \mathbb{E}(m | d\nu(m) d\mu^n_s(\hat{m})) \rangle ds
\]  

(19)

we have

\[
d\hat{S}^n_t = \hat{S}^n_t b(t)dB_t + \hat{S}^n_t \langle f_t(\hat{m}), d\mu^n_t(\hat{m})\rangle^{-1} \int_{\mathcal{M}(B)} (f_t - \nu) \Gamma^n(dt, dv)
\]

Then

\[
\hat{S}^n_t = S_0 \cdot \mathbb{E}_t \left( \int_0^t b(s)dB_s + \int_0^t \langle f_s(\hat{m}), d\mu^n_s(\hat{m})\rangle^{-1} \int_{\mathcal{M}(B)} (f_s - \nu) \Gamma^n(ds, dv) \right)
\]  

(20)
The value of the option at time t is then

\[ C_t^n = \mathbb{E}_{Q^n} \left[ e^{rt} (\tilde{S}_T^n - Ke^{-rt}) + F_t \right]. \] (21)

4 HEDGING EUROPEAN OPTIONS IN THE LIMIT CASE

We have shown in a precedent work that probabilities \( \mathbb{P}^n \) defined on \((\Omega, \mathcal{F})\) where \( \Omega = D([0, T], \mathbb{P} (B) \times \mathbb{R}) \) and \( \mathcal{F}_t \) the natural filtration with respect to processes \( \mu_t^n \) and \( X_t \), converge to a probability limit \( \mathbb{P} \) on \((\Omega, \mathcal{F})\) such that conditional probabilities

\[ \mathbb{P} (\cdot | X_{[0,T]}(\cdot)) = \delta_{\mu([0,T])}, \]

where \( \mu([0,T]) \) satisfies the following functional differential equation

\[ d\mu_t = F(\mu_t)dt, \] (22)

\( F \) is defined in (4) and (5). The price of the risky asset \( S_t \) was defined by \( S_t = S_0, \mathcal{E}_t, \langle \exp\{t, \mu_t\} \rangle \). We will be interested here only on the case \( dX_t = a(t)dt + b(t)dW_t \), with the same condition (7) on \( b \). The discounted price of the asset is given by

\[ \tilde{S}_t = S_0, \mathcal{E}_t, \langle f_t(m), \mu_t \rangle, \] where \( f_t(m) = \exp \{ (\cdot - r)t \} \). \( \tilde{S}_t \) satisfies the following stochastic differential equation

\[ d\tilde{S}_t = \tilde{S}_t (f_t(m), \mu_t)^{-1} \left[ ((m - r)f_t(m), \mu_t) + (f_t, dF(\mu_t)) \right] dt + \tilde{S}_t a(t) dt + \tilde{S}_t b(t) dW_t. \]

Let \( B_t = W_t - \int_0^t \tilde{\beta}_2(s) ds, \) with

\[ \tilde{\beta}_2(t) = \frac{(r - a(t)f_t(m), \mu_t) - (f_t(m), dF(\mu_t))}{b(t)(f_t(m), \mu_t)}, \] (23)

we get \( d\tilde{S}_t = \tilde{S}_t b(t) dB_t. \)
Then there exists a probability $\mathbb{Q}$ (unique with respect to the natural filtration of $W_t$), equivalent to $\mathbb{P}$ defined by

$$
\frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_t} = \mathcal{E}_t \left( \int_0^t \beta_2(s) \, dW_s \right).
$$

(24)

under which the process $B_t$ is a standard brownian motion. Then the discounted price process $\hat{S}_t$ is a $\mathbb{Q}$-martingale

$$
\hat{S}_t = S_0 e^{\int_0^t b(s) \, dB_s - \frac{1}{2} \int_0^t b^2(s) \, ds}.
$$

(25)

The price of Call option at time $t$ is given by

$$
C_t = \mathbb{E}_{\mathbb{Q}} \left[ e^{-r(T-t)} (S_T - K) \right] / \mathcal{F}_t
$$

$$
= \mathbb{E}_{\mathbb{Q}} \left[ \left( S_t e^{\int_t^T b(s) \, dB_s - \frac{1}{2} \int_t^T b^2(s) \, ds} \right) - Ke^{-r(T-t)} \right] / \mathcal{F}_t.
$$

(26)

We know that $\int_0^t b(s) \, dB_s$ is Gaussian, centered with variance $\sigma_t^2 = \int_0^t b^2(s) \, ds$ and $\int_t^T b(s) \, dB_s$ is independent of $\mathcal{F}_t$. Then

$$
C_t = \mathbb{E}_{\mathbb{Q}} \left[ \left( S_t e^{\int_t^T b(s) \, dB_s - \frac{1}{2} \int_t^T b^2(s) \, ds} \right) - Ke^{-r(T-t)} \right],
$$

(27)

$\hat{S}_t$ is $\mathcal{F}_t$-measurable. Hence

$$
C_t = \frac{1}{\sigma_{t,T} \sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( S_t e^{\int_t^T b^2(s) \, ds} \right) \times \exp \left\{ \frac{y^2}{2\sigma_{t,T}^2} \right\} \, dy,
$$

with $\sigma_{t,T}^2 = \int_t^T b^2(s) \, ds$. We have then

$$
C_t = \frac{S_t}{\sigma_{t,T} \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp \left\{ - \frac{1}{2\sigma_{t,T}^2} (y - \sigma_{t,T})^2 \right\} \, dy - \frac{K e^{-r(T-t)}}{\sigma_{t,T} \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp \left\{ - \frac{y^2}{2\sigma_{t,T}^2} \right\} \, dy.
$$

then
\[ C_t = S_t N(d_1) - Ke^{r(T-t)}N(d_2), \]

where

\[ d_1 = \frac{1}{\sigma_t \tau} \left( \log \frac{K}{S_t} + \int_0^T \left( r + \frac{1}{2} b^2(s) \right) ds \right) \]
\[ d_2 = d_1 - \sigma_t \tau. \]

We find in the limit case Black and Scholes formula with parameters depending on time.

5 CONVERGENCE OF RISK NEUTRAL PROBABILITIES

In the two first sections, we have evaluated call options in a system composed of \( n \) agents and have seen that the market has always incomplete by the equation (16), then we was interested to the evaluation of call options in the limit case, and found that the market is complete. It is then very interesting to study the eventualty of the convergence of incomplete market to complete one, when \( n \) goes to infinity, and see if the following diagram is commutative.

![Figure 1. Market convergence diagram](source: The authors.)

We have seen that the probability measure \( \mathbb{P}^n \) converges to the probability measure \( \mathbb{P} \). Having already made changes of probabilities in both cases, we must see now if the possible \( \mathbb{Q}^n \) converge to \( \mathbb{Q} \) when \( n \) goes to infinity. We begin first by studying the tightness of processes \( \mathbb{Q}^n \) in \( \Omega \).

**Proposition 2.** Let the probabilities \( \mathbb{Q}^n \) defined by

\[ \frac{d\mathbb{Q}^n}{d\mathbb{P}^n} / \mathcal{F}_t = \mathcal{E}_t N^n, \]
\[ dN_t^n = \beta_2^n(t)dW_t + \int_{M(B)} \beta_1^n(t, \nu) \tilde{\Gamma}^{(n)}(dv, dt). \] 

(29)

Suppose the conditions (12), (13) and (14) satisfied and for a constant \( C \) we have

\[ \int_0^T \sup_n (\beta_2^n(s))^2 \, ds + \int_0^T \sup_n \sup_{\hat{m} \in B} \left| 1 + \beta_1^n \left( s, \frac{1}{n}(\delta_m - \delta_{\hat{m}}) \right) \right| ds \leq C < +\infty. \] 

(30)

Then the processes \( \mathcal{Q}^n \) are tight.

A first answer to the commutativity of the preceding diagram gives the following result.

**Theorem 2.** Suppose in addition of the condition (16) we have

\[ \sup_{t \leq T} \sup_{m, \hat{m} \in B} \left| \beta_1^n \left( s, \frac{1}{n}(\delta_m - \delta_{\hat{m}}) \right) \right| \xrightarrow{n \to \infty} 0. \] 

(31)

Then the probabilities \( \mathcal{Q}^n \) converge to the limit probability \( \mathcal{Q} \) defined by (24).

The reason of adding (30) is the remarks made during the study of some particular cases in the choice of risk neutral probability. We have seen that there is convergence from incomplete markets to complete ones if we charge macroeconomic components with neglecting microeconomic ones, in the inverse case the convergence is not realized. We are going to give briefly examples concerning some particular cases.

**Example 1.** If we consider the change of probability only on the macroeconomic part (i.e. when \( \beta_1^n(t, \nu) \equiv 0 \)) then we are under the conditions of the theorem and then there is no problem for the convergence.

**Example 2.** (Macroeconomic probability change) Let now the inverse case of example 1, i.e. \( \beta_2^n \equiv 0 \), then the process \( \beta_1^n(t, \nu) \) satisfy the equation

\[ \langle f_t(\tilde{m})[a(t) + \tilde{m} - r], d\mu_t^n(\tilde{m}) \rangle + \langle f_t, dF(\mu_t^n) \rangle + \langle \{(f_t(m) - f_t(\tilde{m})) \left[ 1 + \beta_1^n \left( t, \frac{1}{n}(\delta_m - \delta_{\tilde{m}}) \right) \right], C(m)\rho(m) d\mu_t^n(\tilde{m}) \rangle = 0. \]
Hence $N_t^n = \int_0^t \int_{\mathcal{M}(B)} \beta^n_I(s, \nu) \bar{F}^{(n)}(dv, ds)$.

Letting

$$\int_{\mathcal{M}(B)} \langle f_t, \nu \rangle \bar{L}^{(n)}(dv, dt) = \int_{\mathcal{M}(B)} \langle f_t, \nu \rangle \bar{F}^{(n)}(dv, dt)$$

$$+ \langle ((f_t(m) - f_t(\bar{m}))\beta^n_I(t, \frac{1}{n}(\delta_m - \delta_{\bar{m}})), C(m) d\rho(m) d\mu^n_t(m)) \rangle dt.$$ 

we have

$$d\tilde{S}_t^n = \tilde{S}_t^n(f_t(\bar{m}), d\mu^n_t(\bar{m}))^{-1} \int_{\mathcal{M}(B)} \langle f_t, \nu \rangle \bar{L}^{(n)}(dv, dt) + \tilde{S}_t^n b(t) dW_t.$$ 

and then risk neutral probabilities will be

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}_{/\mathcal{F}_t} = \mathcal{E}_t \left( \int_0^t \int_{\mathcal{M}(B)} \beta^n_I(s, \nu) \bar{F}^{(n)}(dv, ds) \right).$$

In this case $\mathbb{Q}^n$ don't converge to the limit risk neutral probability.

**Example 3.** Suppose now that $\beta^n_I \equiv -1$, then

$$\langle f_t(\bar{m}), d\mu^n_t(\bar{m})\rangle[\alpha(t) + b(t)\beta^n_2(t)] + \langle (\bar{m} - r)f_t(\bar{m}), d\mu^n_t(\bar{m}) \rangle = 0.$$ 

Hence

$$\beta^n_2(t) = \frac{\langle (r - \alpha(t) - \bar{m})f_t(\bar{m}), d\mu^n_t(\bar{m}) \rangle}{b(t)f_t(\bar{m}), d\mu^n_t(\bar{m})},$$

and
\[ N^n_t = -\int_0^t \int_{M(B)} \bar{\Gamma}^{(n)}(dv, ds) + \int_0^t \beta^n_2(s)dW_s. \]

Letting \( B_t = W_t - \int_0^t \beta^n_2(s)ds \)

and

\[ \int_{M(B)} \langle f_t, v \rangle \bar{L}^{(n)}(dv, dt) = \int_{M(B)} \langle f_t, v \rangle \bar{\Gamma}^{(n)}(dv, dt) + \langle f_t, dF(\mu^n_t) \rangle dt. \]

We get

\[ d\tilde{S}^n_t = \tilde{S}^n_t \langle f_t(\tilde{m}), d\mu^n_t(\tilde{m}) \rangle^{-1} \int_{M(B)} \langle f_t, v \rangle \bar{L}^{(n)}(dv, dt) + \tilde{S}^n_t b(t)dB_t. \]

and

\[ \frac{dQ^n_t}{dP^n} / F_t = \mathcal{E}_t \left( \int_0^t \int_{M(B)} \bar{\Gamma}^{(n)}(dv, ds) + \int_0^t \beta^n_2(s)dW_s \right). \]

Remark here that desired convergence is not realized.

We will see in the next section that either if the convergence of risk neutral probability \( Q^n \) to the limit risk neutral probability \( Q \) is not satisfied, the call option price \( C^n_t \) converges to the limit price \( C_t \). It seems very strange but will be explained by the absence of the term

\[ \tilde{S}^n_t \langle f_t(\tilde{m}), d\mu^n_t(\tilde{m}) \rangle^{-1} \int_{M(B)} \langle f_t, v \rangle \bar{L}^{(n)}(dv, dt) \]

in the expression (3.19) of the discounted price of the underlying asset.

6 Stability of Hedging Options

It's easy to conclude, under the conditions of the preceding theorem, that the price of an option on the risky asset with price \( S_t \), is stable under the limit \( n \rightarrow \infty \).
\( \infty \). We note also that Black and Scholes formula is independent of the exponential drift of the underlying asset. This will be seen in formulas (20), (21) and (25), (26) by the absence of a term in \( \beta_1^n \) or \( \beta_2^n \). We find that the options' price \( C_t^n \) converge under general conditions to the limit \( C_t \).

Proposition 3. Let

\[
\hat{N}_t^n = \int_0^t \frac{1}{\langle f_s(\bar{m}), d\mu_s^n(\bar{m}) \rangle} \int_{\mathcal{M}(B)} \langle f_s, v \rangle \bar{U}(n)(dv, ds)
\]

the martingale with respect to \( \mathbb{Q}^n \) given by (28) and (29), \( \beta_1^n, \beta_2^n \) realize the conditions (12), (13), (14), the \( \mathbb{Q}^n \)-martingale \( \mathcal{E}_t \hat{N}_t^n \) converges in law to the constant 1.

7 HEDGING AMERICAN OPTIONS FOR GREAT "N"

7.1 AMERICAN OPTION

An American option buyer can exercise his or her right at any moment until maturity. The decision to exercise or not at a given moment will be based on the facts available at the time. In a continuous-time model built on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}) \) the exercise date is described by a stopping time.

Let \( \mathcal{T} \) be the set of all stopping times with respect to the filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \), and introduce the following subsets of \( \mathcal{T} \):

\[
\mathcal{T}_{t,T} = \left\{ \tau \in \mathcal{T} \mid P_\tau \in [t, T] = 1 \right\}, \quad 0 \leq t \leq T < \infty,
\]

\[
\mathcal{T}_{t,\infty} = \left\{ \tau \in \mathcal{T} \mid P_\tau \in [t, \infty[ = 1 \right\}, \quad t \geq 0.
\]

Theorem 3. Let \( Z = (Z_t)_{0 \leq t} \) be an \( (\mathcal{F}_t)_{0 \leq t} \) adapted \( \mathbb{R} \)-valued càdlàg process that belongs to the class \([D]\), i.e., satisfying \( \mathbb{E} \left( \sup_{0 \leq t \leq T} Z_t \right) < \infty \). The set of random variables \( \{Z_\tau \in \mathcal{T}_0\} \) is uniformly integrable. Then, there exists an \( (\mathcal{F}_t)_{0 \leq t} \)-adapted \( \mathbb{R} \)-valued càdlàg process \( U := (U_t)_{0 \leq t} \) such that \( U \) is the smallest supermartingale which dominates \( Z \), i.e., if \( \hat{U} := (\hat{U}_t)_{0 \leq t} \) is another càdlàg super-
martingale such that for all \( 0 \leq t \leq T, \) \( \hat{U}_t \geq Z_t \) then \( \hat{U}_t \geq U_t \) for any \( 0 \leq t \leq T \). The process \( U \) is called the Snell envelope of \( Z \). Moreover, the following properties hold

1. for any \((\mathcal{F}_t)_{0 \leq t}\)-stopping time for \( t \in [0, T] \) we have:

\[
U_t = \text{ess} \sup_{\tau \in \mathcal{T}_{t,T}} E \left[ \frac{Z_\tau}{\mathcal{F}_\tau} \right], \quad \text{and then } U_T = Z_T.
\]

2. if \( Z \) has only positive jumps, then \( U \) is a continuous process. Furthermore, if \( t \) is an \((\mathcal{F}_t)_{0 \leq t}\)-stopping time and \( \hat{\tau}_t = \{ s \geq t, U_s = Z_s \} \cap T \) then \( \hat{\tau} \) is optimal after \( t \), i.e. \( U_t = E \left[ \frac{U_{\hat{\tau}_t}}{\mathcal{F}_{\hat{\tau}_t}} \right] = E \left[ \frac{Z_{\hat{\tau}_t}}{\mathcal{F}_{\hat{\tau}_t}} \right] = \text{ess} \sup_{\tau \geq t} E \left[ \frac{Z_\tau}{\mathcal{F}_\tau} \right] ; \)

3. if \((Z^n)_{n \geq 0}\) and \( Z \) are cadlag and of class \([D]\) and such that the sequence \((Z^n)_{n \geq 0}\) converges increasingly and pointwisely to \( Z \) then \((U^n Z)_{n \geq 0}\) converges increasingly and pointwisely to \( U Z \); \( U^n Z \) and \( U Z \) are the Snell envelopes of respectively \( Z_n \) and \( Z \). Furthermore, if \( Z \) belongs to \( S^p \) then \( U Z \) belongs to \( S^p \), such that:

a) \( \mathcal{P} \) be the \( \sigma \)-algebra on \([0, T] \times \Omega \) of \((\mathcal{F}_t)_{0 \leq t}\)-progressively measurable sets;

b) \( p > 1 \) be a fixed real constant; \( \mathcal{M}^{p,k} \) is the set of \( \mathcal{P} \)-measurable and \( \mathbb{R}^k \) valued processes \( \omega = (\omega_t)_{0 \leq t \leq T} \) such that \( E \left[ \int_0^T |\omega_s|^p \, ds \right] < \infty \); \( S^p \) is the set of \( \mathcal{P} \)-measurable, \( \mathbb{R} \)-valued, continuous processes \( \omega = (\omega_t)_{0 \leq t \leq T} \) such that \( E \left[ \sup_{0 \leq t \leq T} |\omega_t|^p \right] < \infty \).

**Theorem 4.** There is absence of arbitrage in the market model with trading in an American claim if and only if the price \( V \) is given by the formula, where \( E^* \) is the hope under \( \mathbb{P}^* \), \( \mathcal{T}_{0,T} \) is the set of stopping times with values in \([0, T]\), and \( K \) is the strike. Therefore the price of a call and the put is determined by

\[
C^A_0 = \sup_{\tau \in \mathcal{T}_{0,T}} E^* e^{-rt} [S_T - K], \quad (32)
\]

\[
P^A_0 = \sup_{\tau \in \mathcal{T}_{0,T}} E^* e^{-rt} [K - S_T]. \quad (33)
\]
More generally, the arbitrage price at time $t$ of an American option with reward function $h$ equals

$$C_t^{Am} = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* e^{-r(t-\tau)} [h / \mathcal{F}_\tau],$$

(34)

$$P_t^{Am} = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* e^{-r(t-\tau)} [h / \mathcal{F}_\tau].$$

(35)

**Definition 1.** That is $(H_T)_{0 \leq t \leq T}$ is an adapted process such that $E \left[ \int_0^T H_t^2 \, dt \right] < \infty$, a trading strategy with consumption is defined as an adapted process $\emptyset = (H_t^0, H_t)_{0 \leq t \leq T}$, with values in $\mathbb{R}^2$, satisfying the following properties:

$$\int_0^T |H_t^0| \, dt + \int_0^T H_t^2 \, dt < \infty \text{ a.s.},$$

(36)

$$H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u - C_t.$$  

(37)

For all $t \in [0, T]$, where $(C_t)_{0 \leq t \leq T}$ is an adapted, continuous, non-decreasing process null at $t = 0$; $C_t$ corresponds to the cumulative consumption up to time $t$.

Such that $u \leq t$, $H_t^0$ and $H_t$ are the quantities of riskless asset and risky asset, respectively, held in the portfolio at time $t$.

**Remark 1.**

a) an American option is a process of the form;

$$\left( \psi(t, S_t) \right)_{t \in [0, T]}^\prime$$

where $\psi$ is a convex Lipschitz continuous function on $[0, T] \times \mathbb{R}_+^*$, where $\psi(t, S_t)$ represents the premium obtained by exercising the option at time $t$.

b) an early-exercise strategy is a stopping time on $(\Omega, \mathcal{F}, \mathbb{P}^*, \mathcal{F}_t)$ taking values in $[0,T]$; we denote by $\mathcal{T}_T$ the family of all exercise strategies. We say that $\tau_0 \in \mathcal{T}_{t,T}$ is an optimal strategy if we have that;
Theorem 5. (Theorem of the minimal value of a hedging strategy for an American option in the Black-Scholes model). Let \( u \) be the map from \([0, T] \times \mathbb{R}^+ \) to \( \mathbb{R} \) defined by

\[
\begin{align*}
u(t,x) &= \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^{\mathbb{P}^t}[e^{-r\tau} \psi(x e^{\nu \left(t - \left(\frac{\sigma^2}{2}\right)(\tau - t) + \sigma(W_\tau - W_t)\right)}],
\end{align*}
\]

where \( \mathcal{T}_{t,T} \) is the set of all stopping times with values in \([t, T]\). There exists a strategy \( \emptyset \in \phi^\psi \), such that \( V_t(\emptyset) = u(t, S_t) \), for all \( t \in [0, T] \).

Moreover, for any strategy \( \emptyset \in \phi^\psi \) we have \( V_t(\emptyset) \geq u(t, S_t) \), for all \( t \in [0, T] \). Where \( u(t, S_t) \), as a price for the American option at time \( t \), and \( V_t(\emptyset) \) the value of the portfolio at time \( t \). \( \phi^\psi \) the set of all trading strategies with consumption hedging the American option defined by \( h(t) = \psi(S_t) \).

7.2 RISK-NEUTRAL PROBABILITIES

Our reference space is \((\Omega, \mathcal{F}, \mathbb{P}^n)\) such that \( \Omega = D([0, T], \mathcal{P}(B) \times \mathbb{R}) \), \( \mathcal{P}(B) \) is endowed with a weak -*-topology, \( \mathbb{P}^n \) is the historic probability and we suppose that the filtration \( \mathcal{F} \) is minimal relative to the processes \( \mu_t^n \) and \( X_t \). Then we have the following result:

**Proposition 4.** The risk-neutral probabilities \( \mathbb{Q}^n \) (in the system of \( n \) agents) are defined by

\[
d\mathbb{Q}^n / \mathcal{F}_t = \mathcal{E}_t N^n. d\mathbb{P}^n / \mathcal{F}_t,
\]

with

\[
N_t^n = \int_0^t \int_{\mathcal{M}(\mathbb{B})} \beta_t^n(s, \nu) \bar{\Gamma}^{(n)}(d\nu, ds) + \int_0^t \beta_t^n(s)dW_s,
\]

the increasing process with \( N_t^n \) is
\(\langle N^n, N^n \rangle_t = \int_0^t (\beta^n_s)^2 \, ds + n \int_0^t (\langle \beta^n_s \rangle^2 \left( s, \frac{1}{n} (\delta_m - \delta_m) \right), C(m) \, d\rho(m)d\mu^n_t(m)) \, ds, \quad (43)\)

so the process \(\tilde{S}_t^n\) is a \(\mathbb{Q}^n\)–super-martingale.

**Proof.** We must prove that \(\tilde{S}_t^n\) is a super-martingale under \(\mathbb{Q}^n\), knowing that \(\mathcal{E}_t(N^n)\) a local martingale strictly positive verifying

\[E(\mathcal{E}_t(N^n)) = 1,\]

and

\[d\mathcal{E}_t(N^n) = \mathcal{E}_t N^n \, dN^n_t, \text{ under } \mathbb{P}^n,\]

as well as \(N^n_t\) is a continuous local martingale null in 0, hence the following question: Does \(\tilde{S}_t^n \mathcal{E}_t N^n\) is a super-martingale under \(\mathbb{Q}^n\)?

We consider \(\tau_n\) a stopping time whatever, \(\delta_t\) is a step such as \(t \in [0, T] = \{0, \delta_t, 2\delta_t, \ldots, T\}\), we know that the indicator function 1 is an integrable convex function verifying \(1 = 1_{t<\tau, n} + 1_{t \geq \tau, n}\), the two indicators are \(\mathcal{F}_t\) – measurable, we suppose that \(\mathcal{E}_t N^n\) and \(S^n_t\) the two processes are independent, and that

\[E \left[ \exp \left( \frac{1}{2} \langle N^n, N^n \rangle_T \right) \right] < +\infty \text{ condition of Novikov}, \quad (44)\]

we have

\[
\begin{align*}
E \left[ \tilde{S}^n_{t \wedge \tau_n} \mathcal{E}_{t \wedge \tau_n} N^n - \tilde{S}^n_{(t-\delta_t) \wedge \tau_n} \mathcal{E}_{(t-\delta_t) \wedge \tau_n} N^n / \mathcal{F}_{t-\delta_t} \right] \\
= 1_{t<\tau_n} E \left[ \tilde{S}^n_{t \wedge \tau_n} \mathcal{E}_{t \wedge \tau_n} N^n - \tilde{S}^n_{(t-\delta_t) \wedge \tau_n} \mathcal{E}_{(t-\delta_t) \wedge \tau_n} N^n / \mathcal{F}_{t-\delta_t} \right] \\
+ 1_{t \geq \tau_n} E \left[ \tilde{S}^n_{t \wedge \tau_n} \mathcal{E}_{t \wedge \tau_n} N^n - \tilde{S}^n_{(t-\delta_t) \wedge \tau_n} \mathcal{E}_{(t-\delta_t) \wedge \tau_n} N^n / \mathcal{F}_{t-\delta_t} \right] \\
= E \left[ 1_{t<\tau_n} \left( \tilde{S}^n_{t \wedge \tau_n} \mathcal{E}_{t \wedge \tau_n} N^n - \tilde{S}^n_{(t-\delta_t) \wedge \tau_n} \mathcal{E}_{(t-\delta_t) \wedge \tau_n} N^n / \mathcal{F}_{t-\delta_t} \right) \right] \\
+ E \left[ 1_{t \geq \tau_n} \left( \tilde{S}^n_{t \wedge \tau_n} \mathcal{E}_{t \wedge \tau_n} N^n - \tilde{S}^n_{(t-\delta_t) \wedge \tau_n} \mathcal{E}_{(t-\delta_t) \wedge \tau_n} N^n / \mathcal{F}_{t-\delta_t} \right) \right]
\end{align*}
\]
\[
E[1_{t<\tau_n}(\bar{S}^n_t \mathcal{E}_t N^n - \bar{S}^n_{(t-\delta_t)} \mathcal{E}_{(t-\delta_t)} N^n / \mathcal{F}_{t-\delta_t})]
\]

\[
+ E[1_{t\geq\tau_n}(\bar{S}^n_t \mathcal{E}_t N^n - \bar{S}^n_{t\tau_n} N^n / \mathcal{F}_{t-\delta_t})]
\]

\[
= E[1_{t<\tau_n}(\bar{S}^n_t \mathcal{E}_t N^n / \mathcal{F}_{t-\delta_t})] - \bar{S}^n_{(t-\delta_t)} \mathcal{E}_{(t-\delta_t)} N^n, \quad \bar{S}^n_{(t-\delta_t)} \mathcal{E}_{(t-\delta_t)} N^n \mathcal{F}_{t-\delta_t} \text{measurable}
\]

\[
= 1_{t<\tau_n}[E(\bar{S}^n_t / \mathcal{F}_{t-\delta_t}) \times E(\mathcal{E}_t N^n / \mathcal{F}_{t-\delta_t})] - \bar{S}^n_{(t-\delta_t)} \mathcal{E}_{(t-\delta_t)} N^n.
\]

(Since \(\bar{S}^n_t\) and \(\mathcal{E}_t N^n\) are conditionally independent and \(\mathcal{E}_t N^n\) is a martingale)

\[
= \mathcal{E}_{(t-\delta_t)}[1_{t<\tau_n}E(\bar{S}^n_t / \mathcal{F}_{t-\delta_t})] - \bar{S}^n_{(t-\delta_t)} \mathcal{E}_{(t-\delta_t)} N^n,
\]

\[
= \mathcal{E}_{(t-\delta_t)}[1_{t<\tau_n}E(\bar{S}^n_t - \bar{S}^n_{(t-\delta_t)}/ \mathcal{F}_{t-\delta_t})].
\]

We prove now that \([1_{t<\tau_n}E(\bar{S}^n_t - \bar{S}^n_{(t-\delta_t)}/ \mathcal{F}_{t-\delta_t})]\) is a super-martingale, so we have to prove that \(E(\bar{S}^n_t / \mathcal{F}_{t-\delta_t}) \leq \bar{S}^n_{(t-\delta_t)}\)?

Since \(S_0\) is \(\mathcal{F}_0\)-measurable, \(\mathcal{E}_t X\) is a martingale and \(\mathcal{E}_t X\) and \(\langle f_t(\bar{m}), d\mu^n_t(\bar{m}) \rangle\) are independent, we have,

\[
1_{t<\tau_n}E(S_0 \cdot \mathcal{E}_t X \cdot \langle f_t(\bar{m}), d\mu^n_t(\bar{m}) \rangle / \mathcal{F}_{t-\delta_t})
\]

\[
= 1_{t<\tau_n}S_0 E(\mathcal{E}_t X \cdot \langle f_t(\bar{m}), d\mu^n_t(\bar{m}) \rangle / \mathcal{F}_{t-\delta_t}),
\]

\[
= 1_{t<\tau_n}S_0 [E(\mathcal{E}_t X / \mathcal{F}_{t-\delta_t}) \cdot E(\langle f_t(\bar{m}), d\mu^n_t(\bar{m}) \rangle / \mathcal{F}_{t-\delta_t})],
\]

\[
= 1_{t<\tau_n}S_0 E_{t-\delta_t} X \cdot E(\langle f_t(\bar{m}), d\mu^n_t(\bar{m}) \rangle / \mathcal{F}_{t-\delta_t}),
\]

\[
= 1_{t<\tau_n}S_0 E_{t-\delta_t} X \cdot E \left( \lim_{n \to +\infty} (f_{t\wedge T_n}(\bar{m}), d\mu^n_{t\wedge T_n}(\bar{m})) / \mathcal{F}_{t-\delta_t} \right),
\]

\[
\leq S_0 E_{t-\delta_t} X \cdot \lim_{n \to +\infty} 1_{t<\tau_n}E(\langle f_{t\wedge T_n}(\bar{m}), d\mu^n_{t\wedge T_n}(\bar{m}) \rangle / \mathcal{F}_{t-\delta_t}),
\]

\[
\leq S_0 E_{t-\delta_t} X \cdot \lim_{n \to +\infty} 1_{t<\tau_n} (f_{t-\delta_t} X \wedge T_n(\bar{m}), d\mu^n_{t-\delta_t} X \wedge T_n(\bar{m})),
\]
\[ \leq S_0 \mathcal{E}_{t-\delta_t} X_t (f(t-\delta_t)(\tilde{m}), d\mu_{t-\delta_t}(\tilde{m})). \]

Hence

\[ E(\tilde{S}_t^n / \mathcal{F}_{t-\delta_t}) \leq \tilde{S}_{t-\delta_t} \Rightarrow E(\tilde{S}_t^n - \tilde{S}_{t-\delta_t} / \mathcal{F}_{t-\delta_t}) \leq 0, \]

And since \( \mathcal{E}_{t-\delta_t} \mathcal{N}^n \) is a positive variable according to the conditions of Novikov, hence \( \forall \tau_n \) we have

\[ E[\tilde{S}_t^n \mathcal{E}_{t\wedge \tau_n} N^n - \tilde{S}_{t-\delta_t} \mathcal{E}_{t\wedge \tau_n} N^n / \mathcal{F}_{t-\delta_t}] \leq 0, \]

\[ \Rightarrow E[\tilde{S}_t^n \mathcal{E}_{t\wedge \tau_n} N^n / \mathcal{F}_{t-\delta_t}] \leq \tilde{S}_{t-\delta_t} \mathcal{E}_{t\wedge \tau_n} N^n. \]

Which is a super-martingale under \( \mathbb{Q}^n \). From where, \( \tilde{S}_t^n \) is a \( \mathbb{Q}^n \)-super-martingale.

Then, the value of the American option at time \( t \) is

\[ (C_t^n)^{Am} = \text{ess sup}_{\tau \in \mathcal{T}_t} E_{\mathbb{Q}^n} \left[ e^{-r(t-\tau)}(\tilde{S}_T^n - K e^{-r\tau})_+ / \mathcal{F}_t \right]. \]

8 HEDGING AMERICAN OPTION IN THE LIMIT CASE

We have shown previously that probabilities \( \mathbb{P}^n \) defined on \( (\Omega, \mathcal{F}) \) where \( \Omega = D([0, T], \mathbb{P} (\mathcal{B}) \times \mathbb{R}) \), and \( \mathcal{F}_t \) the natural filtration with respect to process \( \mu_t^n \) and \( X_t \) converge to a probability limit \( \mathbb{P} \) on \( (\Omega, \mathcal{F}) \) and that there exists a unique probability \( \mathbb{Q} \) with respect to the natural filtration of \( W_t \) equivalent to \( \mathbb{P} \) defined by

\[ \frac{d\mathbb{Q}}{d\mathbb{P}} / \mathcal{F}_t = \mathcal{E}_t \left( \int_0^t \tilde{\beta}_2 (s) dW_s \right), \]  

(45)

under which the process \( B_t \) is a standard Brownian motion. Thereby the actual price of the risky asset is a \( \mathbb{Q} \)-super-martingale.

\[ \tilde{S}_t = S_0 \exp \left\{ \int_0^t b(s) dB_s - \frac{1}{2} \int_0^t b^2 (s) ds \right\}. \]

(46)
The price of the American call option at time $t$ is given by

$$C_t^{Am} = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E_Q \left[ e^{-r(\tau-t)} (S_\tau - K)_+ / \mathcal{F}_t \right]$$

$$= \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E_Q \left[ \left( S_T \exp \left\{ \int_t^T b(s) dB_s - \frac{1}{2} \int_t^T b^2(s) ds \right\} - ke^{-r(\tau-t)} \right)_+ / \mathcal{F}_t \right].$$

With the same notations as in section 6, we have the value of the option at time $t$ defined by

$$C_t^{Am} = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E_Q \left[ \left( S_T \exp \left\{ \int_t^T b(s) dB_s - \frac{1}{2} \int_t^T b^2(s) ds \right\} - ke^{-r(\tau-t)} \right)_+ \right]. \quad (47)$$

according to the theorem 5., we have

$$C_t^{Am} = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[ \left. e^{-r(\tau-t)} \left( x \exp \left( r - \left( \frac{\sigma^2}{2} \right) (\tau - t) + \sigma (W_\tau - W_t) \right) \right) \right| \right. \left. \mathcal{F}_t \right],$$

$$= \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[ e^{-r(\tau-t)} \left( x \exp \left( r - \left( \frac{\sigma^2}{2} \right) (\tau - t) + \sigma (W_\tau - W_t) \right) - K \right)_+ \right],$$

$$= \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[ x \exp \left( \sigma (W_\tau - W_t) - \left( \frac{\sigma^2}{2} \right) (\tau - t) \right) - ke^{-r(\tau-t)} \right]_+, \quad (48)$$

under $\mathbb{P}^*$, $(W_\tau - W_t)$ is a centered Gaussian Variable with variance $(\tau - t)$, therefore

$$C_t^{Am} = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[ x \exp \left( \sigma \sqrt{\tau - t} g - \left( \frac{\sigma^2}{2} \right) (\tau - t) \right) - ke^{-r(\tau-t)} \right], \quad (49)$$

g is a standard Gaussian distribution. $S_0$ in accordance with the formula (49) and if we put $x = S_t$ and $\sigma = b(s)$ we find ourselves with the famous formula of the Black and Scholes.
9 CONCLUSION

In this study, we have employed the financial market model proposed by Remita and Eisele, characterized by a large number of agents denoted as "n". Our primary focus has been on the evaluation of American options within this framework. Particularly noteworthy is our demonstration of the super-martingale property exhibited by the actual price of the risky asset $\tilde{S}_t^n$ under the new probability measure $\mathbb{Q}^n$, a crucial step in assessing these options. Furthermore, we have explored the limit case, where a specific value of the asset volatility $\sigma$ leads us to the familiar Black-Scholes formula, providing a broader perspective on option pricing dynamics within the context of Remita and Eisele's model.
REFERENCES


