Nonparametric conditional mode estimate under doubly truncated model

Estimativa de modo condicional não paramétrico sob modelo duplamente truncado

DOI: 10.54021/seesv5n1-133

Recebimento dos originais: 07/05/2024
Aceitação para publicação: 28/05/2024

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ABSTRACT
Conditional mode function is often used in various fields such as statistics, data analysis, and machine learning to understand the distribution of data and make decisions based on specific conditions or criteria. In this paper, we propose a non-parametric kernel estimator of the conditional mode function, when the variable of interest is subject to random doubly truncation. We prove the suggested estimator's strong consistency with a rate and state its asymptotic normality under some regularity assumptions. Our results are based on special mathematical techniques such as the iterative non-parametric maximum likelihood estimators (NPMLE) of the distribution function and the Vapnik-Cervonenkis classes for which uniform exponential inequalities are available. This study will be a valuable resource for scholars and practitioners interested in non-parametric kernel estimation methods for doubly truncated data.

Keywords: asymptotic normality, Kernel estimator, mode, rate of convergence, truncation.
sugerido com uma taxa e declaramos sua normalidade assintótica sob algumas suposições de regularidade. Nossos resultados baseiam-se em técnicas matemáticas especiais, como os estimadores iterativos não paramétricos de máxima verossimilhança (NPMLE) da função de distribuição e as classes de Vapnik-Cervonenkis para as quais existem desigualdades exponenciais uniformes. Esse estudo será um recurso valioso para acadêmicos e profissionais interessados em métodos de estimativa de kernel não paramétricos para dados duplamente truncados.

**Palavras-chave:** normalidade assintótica, estimador de Kernel, modo, taxa de convergência, truncamento.

1 INTRODUCTION

The estimation of the conditional mode has a long history and has been studied by many authors in the statistics literature. For example, Parzen [11] established weak consistency and asymptotic normality for the i.i.d. case and the strong consistency was obtained by [9] and [11]. Ryzin [3] derived the asymptotic normality under weaker conditions than those imposed by Parzen [11].

Chernoff [1] studied the estimator of the mode defined as the center of that interval which contains the most observations. Collomb et al. [2] investigate asymptotic consistency of the kernel estimate of the conditional mode and give some insight into situations. Samanta and Thavaneswaran [13] showed that the kernel estimator of the conditional mode function is consistent and asymptotically normally distributed. Ezzahrioui [4] consider the estimation of the conditional mode when the co-variables take values in some abstract function space and showed that under some regularity conditions, the kernel estimate of the conditional mode is asymptotically normally distributed, under left truncation model [10] constructed a nonparametric kernel estimator of the conditional mode function for the left-truncation model and established the rate of the strong uniform consistency of the estimate as well as the asymptotic normality of the conditional mode. Vieu [14] obtained a rate of convergence for both local and global estimates of the mode function.

To our knowledge, the problem of estimating the conditional mode estimator under doubly truncation model has not been addressed in the statistics literature. This is the central object of interest of this paper.

This paper is organized as follows: in section 2 we define some important and useful results in the random doubly truncation model, then we define the kernel
conditional mode estimator under random double truncation in section 3. Assumption and main results are given in section 4 with asymptotic normality of the suggested estimator. The proofs of the main results are proposed to section 5, finally main conclusion and a final discussion are given in section 6.

2 FRAMEWORK OF RANDOMLY DOUBLY TRUNCATION MODEL

Let \((Y_i)_{1 \leq i \leq n}\) be an i.i.d sample of r.v with continuous distribution function \((df) F\) and density \(f\). Let \(X\) be a r.v for any \(x\), we note by \(f(.\mid x)\) the conditional probability density function of \(Y\) given \(X = x\). For any \(y\), \(f(y\mid x) = \frac{f(x,y)}{l(x)}\), where \(f(x,y)\) is the joint probability density function of \((X,Y)\) and \(l(x)\) is the marginal density of \(X\).

We assume that \(f(y\mid x)\) has a unique mode \(\theta(x)\), and the conditional mode of \(Y\) given \(X = x\), is defined by:

\[
\theta(x) = \arg\max_{y \in \mathbb{R}} f(y\mid x)
\]  

(1)

The only thing we can see because of the random double truncation is that \((X^*,Y^*)\) when \(U^* \leq Y^* \leq V^*\) where \((U^*, V^*)\) are also observed.

On the contrary, when \(U^* \leq Y^* \leq V^*\) is violated nothing is seen. We assume that the truncation times, as is customary with random truncation, are independent of \((X^*,Y^*)\). Let \((Y_1, X_1, U_1, V_1), \ldots, (Y_n, X_n, U_n, V_n)\) be the observed sample, these are iid data with the same distribution as \((X^*, Y^*, U^*, V^*)\) given \(U^* \leq Y^* \leq V^*\).

In the doubly truncated setup, this relative probability of observing \((X^*, Y^*) = (x, y)\) is given by \(G(y) = P(U^* \leq y \leq V^*)\), since \((X^*, Y^*)\) and \((U^*, V^*)\) are independent. This function \(G\) can be estimated from the data by maximum likelihood principles see [7].

For any distribution \(W\) denote the right and left endpoints of its support by

\[
a_w = \inf\{t: W(t) > 0\} \quad \text{and} \quad b_w = \inf\{t: W(t) = 1\}.
\]
Let \( H_1 (u) = H (u, \infty) \) and \( H_2 (v) = H (-\infty, v) \) the marginal df's of \( U^* \) and \( V^* \) respectively. Pointed out that \( F \) and \( H \) are both completely identifiable only if

\[
a_{H_1} \leq a_F \leq a_{H_2} \quad \text{and} \quad b_{H_1} \leq b_F \leq b_{H_2}. \quad (2)
\]

Let \( F(. / x) \) be the conditional df of \( Y^* \ given \ \ X^* = x \), and let

\[
\alpha(x) = P(U^* \leq Y^* \leq V^* / X^* = x) = \int_{-\infty}^{+\infty} G(t) F(dt / x) \quad (3)
\]

be the conditional probability of no truncation. It is assumed that \( G(t) > 0 \). Let \( F^*(. / x) \) be the observable conditional df, that

\[
F^*(. / x) = P(Y_1 \leq y / X_1 = x) \quad (4)
\]

we have

\[
F_n(y / x) = \alpha_n(x) \int_{-\infty}^{y} G_n(t)^{-1} F_n^*(dt / x) \quad \text{for every} \quad y, \quad (5)
\]

and \( F^*(y / x) \) is the distribution function df of the observed \( Y_i's \),

\[
F^*(y / x) = \alpha(x)^{-1} \int_{-\infty}^{y} G(t) F(dt / x). \quad (6)
\]

Recall that the conditional probability estimator of no truncation is represented by

\[
\alpha_n(x) = \left( \int_{a_F}^{b_F} G_n(t)^{-1} F_n^*(dt / x) \right)^{-1} \quad (7)
\]
3 THE ESTIMATORS

In this section we recall some results and then define our mode estimator. We denote by \((U^*, V^*)\) the pair of truncating variables, with joint distribution function \(H\), so \(Y^*\) is observed only when \(U^* \leq Y^* \leq V^*\). Besides a vector of covariates \(X^*\) is attached to \(Y^*\) and, naturally, it will be available to the researcher only when \(Y^*\) is not truncated. In light of the sampling data is indicated by \((X_i, Y_i, U_i, V_i), 1 \leq i \leq n\); given that \(U^* \leq Y^* \leq V^*\) these are iid copies with the conditional distribution of \((X, Y, U, V)\). It is believed that \((U^*, V^*)\) is independent of \((X^*, Y^*)\).

In this essay, we'll suppose that

\[
a_F = \inf\{t: F(t) > 0\} \quad \text{and} \quad b_F = \inf\{t: F(t) = 1\},
\]

Where:

\(a_F\) and \(b_F\) denote the left and right endpoints of the distribution \(F\). If no truncation is present, it is well known that the kernel estimator of the conditional mode function of \(Y\) given \(X = x\) is defined by the following equation

\[
\hat{\theta}_n(x) = \arg\max_{y \in \mathbb{R}} \hat{f}_n(y/x)
\]

(8)

with

\[
\hat{f}_n(y/x) = \frac{\hat{f}_n(x,y)}{\hat{l}_n(x)}
\]

(9)

where

\[
\hat{f}_n(x,y) = \frac{\alpha_n}{nh^2} \sum_{i=1}^{n} G_n(X_i)^{-1} K \left( \frac{x-X_i}{h} \right) L \left( \frac{y-Y_i}{h} \right)
\]

(10)

and

\[
\hat{l}_n(x) = \frac{\alpha_n}{nh} \sum_{i=1}^{n} G_n(X_i)^{-1} K \left( \frac{x-X_i}{h} \right)
\]

(11)
Note that the estimate $\hat{\theta}_n(x)$ is not necessarily unique and our results are valid for any chosen value satisfying (8). We can express our preference by taking

$$
\hat{\theta}_n(x) = \inf \left\{ a_F \leq t \leq b_F \text{ such that } \hat{f}_n(t/x) = \sup_{a_F \leq y \leq b_F} \hat{f}_n(y/x) \right\}.
$$

(12)

Some additional notations are required, to formulate our results, for all kernel $K$; $K^{(j)}$ denotes the $j$-order derivative of $K$. For $(i,j) \in \mathbb{N}^2$, set

$$
f^{(i,j)}(x,y) = \frac{\partial^{(i+j)}}{\partial x^i \partial y^j} f(x,y),
$$

(13)

and for $j \geq 1$,

$$
\hat{f}_n^{(0,j)}(x,y) = \frac{\partial^j}{\partial y^j} \hat{f}_n(x,y) = \frac{a_n}{nh^{(2+j)}} \sum_{i=1}^{n} G_{n}^{-1}(Y_i) K \left( \frac{x-X_i}{h} \right) L^{(j)} \left( \frac{y-Y_i}{h} \right).
$$

(14)

Define $\Omega_0 = \{ x \in \mathbb{R} : l(x) > 0 \}$ and let $\Omega$ be a compact subset of $\Omega_0$.

Consider now the following regularity assumptions:

### 3.1 Assumption and Main Results

(A1) The joint density $f(.,.)$ is differentiable up to order 4 and $\sup_{x,y} |f^{(i,j)}(x,y)|$ exist and are bounded for $1 \leq i + j \leq 4$

(A2) $f^{(0,2)}(x,y)$ and $f^{(2,0)}(x,y)$ are continuous.

(A3) the marginal density $l(.)$ has a continuous second derivative.

(A4) $K$ is a continuous and positive function, $\int tK(t) = 0$, $\int tL^{(1)}(t)dt = 0$ and $K(.)L^{(1)}(.) < \infty$

(A5) $K$ is Lipchitzian, differentiable and bounded and $L$ is three times differentiable and bounded.

(A6) $G(.)$ is continue, $f(./.)$ and $G^{-1}f(./.)$ are twice continuously differentiable.

(A7) The bandwidth $h_n = h$ satisfy $h \to 0$, $\frac{\log n}{nh^7} \to 0$ and $nh^7 \to 0$ as $n \to \infty$.  

(A8) $L$ and $L^{(2)}$ are Lipchitzian, $\mu_2(K) = \int t^2 K(t)dt < \infty$, $R(K) = \int K^2(t)dt < \infty$, $\mu_2(L) = \int t^2 L(t)dt < \infty$, $R(L) = \int L^2(t)dt < \infty$.

Remark 3.1 Since there are only a limited number of variations in $L$, its first derivative, $L^{(1)}$, is integrable. Consequently, $K(.)L^{(1)}(.)$ is integrable. Additionally, the existence of the asymptotic variance term is guaranteed by the integrability of $[K(.)L^{(1)}()]^2$ Proposition 3.1 illustrates our first result, which is the almost sure uniform convergence with a rate of a conditional probability density. Theorem 3.1 shows our second result, which is the almost sure uniform convergence of the conditional mode estimator.

3.2 CONSISTENCY

In order to analyze the asymptotic properties of our estimator we introduce the artificial estimator based on the true $\alpha$ and $G$.

$$\tilde{f}_n(x,y) = \frac{\alpha}{nh^3} \sum_{i=1}^n G(Y_i)^{-1} K\left(\frac{x - X_i}{h}\right) L\left(\frac{y - Y_i}{h}\right).$$ (17)

Proposition 3.1 Suppose that assumptions (A1) - (A5) hold then,

$$\sup_{x \in \Omega} \sup_{\alpha \neq x \neq b} \left| \tilde{f}_n (y/x) - f(y/x) \right| = O\left\{ \max \left( \frac{\log n}{nh^2}, h^2 \right) \right\}. \quad (16)$$

Theorem 3.1 Under the assumptions of Proposition 4.1, if the conditional density satisfies $\sup_{x \in \Omega} f^{(2)}(\theta(x)/x) < 0$, we have

$$\sup_{x \in \Omega} |\tilde{\theta}_n(x) - \theta(x)| = O\left\{ \max \left( \frac{\log n}{nh^2}^{1/4}, h \right) \right\}. \quad (17)$$

Remark 3.2. The assumption of uniform negativity on the second derivative of the conditional density in Theorem 3.1. implies the uniform uniqueness of the conditional mode, i.e.: $\forall \varepsilon > 0$ $\exists \lambda > 0$, $\forall \eta : I \rightarrow \mathbb{R}$, $\sup_{x \in \Omega} |\theta(x) - \eta(x)| \geq \varepsilon \Rightarrow \sup_{x \in \Omega} |f(\theta(x)/x) - f(\eta(x)/x)| \geq \lambda$. 
3.3 ASYMPTOTIC NORMALITY

Now suppose that the density function \( f(y/x) \) is unimodal at \( \theta(x) \). Then by assumption (A1) we have \( f^{(1)}(\theta(x)/x) = 0 \) and we assume that \( f^{(2)}(\theta(x)/x) < 0 \).

Similarly, we have

\[
\hat{f}_n^{(1)}(\hat{\theta}_n(x)/x) = 0 \text{ and } \hat{f}_n^{(2)}(\hat{\theta}_n(x)/x) < 0.
\] (18)

If \( \hat{\theta}_n(x) \) is the mode of \( \hat{f}_n(./x) \).

We obtain using a Taylor expansion

\[
\hat{f}_n^{(1)}(\hat{\theta}_n(x)/x) = \hat{f}_n^{(1)}(\theta(x)/x) + (\hat{\theta}_n(x) - \theta(x)) \hat{f}_n^{(2)}(\hat{\theta}_n(x)/x) = 0,
\] (19)

where \( \tilde{\theta}_n(x) \) is between \( \hat{\theta}_n(x) \) and \( \theta(x) \) and using (5) we can write

\[
\hat{\theta}_n(x) - \theta(x) = -\frac{\hat{f}_n^{(0,1)}(x,\theta(x))}{\hat{f}_n^{(0,2)}(x,\tilde{\theta}_n(x))}.
\] (20)

In order to establish the asymptotic normality of \( \theta_n(x) \), we show that the numerator in (20), suitably normalized is asymptotically normally distributed, and that the denominator converges in probability to \( f^{(0,2)}(x,\theta(x)) \). The result is given in the following Theorem.

**Theorem 3.2** Suppose that assumptions \( (A2), (A4), (A5), (A7) \) and \( (A8) \) holds, we have

\[
(nh^4)^{\frac{1}{2}} \left( \hat{\theta}_n(x) - \theta(x) \right) \xrightarrow{D} N(0, \sigma^2(x)),
\]

where \( \xrightarrow{D} \) denote the convergence in distribution,

\[
\sigma^2(x) = a^2 f(x, \theta(x)) \int_{\mathbb{R}^2} \frac{K^2(r)}{G(r)} \left[ L^{(1)}(s) \right]^2 dr ds.
\]

**Theorem 3.3.** Subject to the following assumptions \( (A4), (A5), (A6) \) and \( (A8) \)
when $n \to \infty$ we have,

$$E(\hat{f}_n(y/x)) = f(y/x) + \frac{h^2}{2} \mu_2(K) \frac{\partial^2}{\partial x^2} f(y/x) + \frac{h^2}{2} \mu_2(L) \frac{\partial^2}{\partial y^2} f(y/x) + O(h^2) \quad (21)$$

$$Var(\hat{f}_n(y/x)) = \frac{\alpha G(y)^{-1} f(y/x) R(K) R(L)}{n h^2 f(x)} + O\left(\frac{1}{nh^2}\right) \quad (22)$$

Theorem 3.3 shows that double truncation has no effect on variance or bias. We will now look at the overall error of $\hat{f}_n(y/x)$, This can be measured by the mean square error $MSE$. Adding the variance (21) to the squared bias (22) gives the asymptotic mean square error

$$AMSE = [E(\hat{f}_n(y/x)) - f(y/x)]^2 + Var(\hat{f}_n(y/x))$$

$$= \left[\frac{h^2}{2} \mu_2(K) \frac{\partial^2}{\partial x^2} f(y/x) + \frac{h^2}{2} \mu_2(L) \frac{\partial^2}{\partial y^2} f(y/x)\right]^2 + \frac{f(y/x) R(K) R(L)}{n h^3 f(x)} \quad (23)$$

The integrated mean square error (MISE) is obtained by taking the double integral with respect to $x$ and $y$ of the weighted mean square error (MSE) formed by the product of (23) with $f(x)$, under regularity, we have from the previous results the following asymptotic expression:

$$MISE = A_1 h^4 + A_2 h^4 - A_3 h^4 + \frac{1}{n h^3} A_4 \quad (24)$$

with

$$A_1 = \frac{1}{4} \iint \mu_2(K)^2 \left[\frac{\partial^2}{\partial x^2} f(y/x)\right]^2 f(x)dx dy;$$

$$A_2 = \frac{1}{4} \iint \mu_2(L)^2 \left[\frac{\partial^2}{\partial y^2} f(y/x)\right]^2 f(x)dx dy;$$
\[ A_3 = \iint 2 \left( \frac{\mu_2(K)}{2} \frac{\partial^2 f(y/x)}{\partial x^2} \right) \left( \frac{\mu_2(L)}{2} \frac{\partial^2 f(y/x)}{\partial y^2} \right) f(x) dx dy; \]

\[ A_4 = \iint \frac{f(y/x) R(K) R(L)}{n h^3 f(x)} f(x) dx dy; \]

By differentiating the expression (24) and setting it to zero, we obtain the width of the asymptotic optimum window

\[ h_{MISE} = \left[ \frac{f(y/x) R(K) R(L)}{f(x) \left( \mu_2(K)^2 \left( \frac{\partial^2 f(y/x)}{\partial x^2} \right)^2 + \mu_2(L)^2 \left( \frac{\partial^2 f(y/x)}{\partial y^2} \right)^2 \right) + \mu_2(K) \mu_2(L) \left( \frac{\partial^2 f(y/x)}{\partial x \partial y} \right)^2} \right]^{1/7} n^{-1/7}. \tag{25} \]

4 AUXILIARY RESULTS AND PROOFS

**Lemma 4.1.** Under assumptions (A3), (A4) and (A5), we have

\[ \sup_{x \in \Omega} |\hat{l}_n(x) - l(x)| = O \left( \max \left( \frac{\log n}{nh^2}, h^2 \right) \right). \tag{26} \]

**Proof.** We define \( \hat{l}_n(x) = \frac{an}{nh} \sum_{i=1}^{n} \frac{1}{d_n(X_i)} K \left( \frac{x - X_i}{h} \right) \)

\[ \sup_{x \in \Omega} |\hat{l}_n(x) - l(x)| \leq \sup_{x \in \Omega} |\hat{l}_n(x) - E(\hat{l}_n(x))| + \sup_{x \in \Omega} |E(\hat{l}_n(x)) - l(x)| \]

Using Taylor expansion, we have under the assumptions (A3)-(A5)

\[ \sup_{x \in \Omega} |E(\hat{l}_n(x)) - l(x)| = O(h^2) \tag{27} \]

Moreover, Assumption (A1) provides a proof similar to that of Proposition 3.1, which gives

\[ \sup_{x \in \Omega} |\hat{l}_n(x) - E(\hat{l}_n(x))| = O \left( \left( \frac{\log n}{nh^2} \right)^{1/2} \right). \tag{28} \]
This concludes the proof.

**Lemma 4.2.** Under assumptions (A4) and (A5) we have

\[
\sup_{x \in \Omega} \sup_{a_F \leq y \leq b_F} |\tilde{f}_n(x, y) - f(x, y)| = O(h_n^2).
\]  

(29)

**Proof.** We define \( V_n = \sup_{x \in \Omega} \sup_{a_F \leq y \leq b_F} |\tilde{f}_n(x, y) - f(x, y)| \). We have,

\[
\tilde{f}_n(x, y) - f(x, y) = \frac{1}{h^2} \int_{-\infty}^{+\infty} K \left( \frac{x-X_i}{h} \right) L \left( \frac{y-Y_i}{h} \right) [\tilde{f}_n(du, dv) - F(du, dv)].
\]  

(30)

Using Fubini's theorem, we obtain by multiple integration by parts

\[
V_n = \sup_{x \in \Omega} \sup_{a_F \leq y \leq b_F} \left\{ \frac{1}{h^2} \int_{-\infty}^{+\infty} K \left( \frac{x-X_i}{h} \right) L \left( \frac{y-Y_i}{h} \right) f(du, dv) \right\}
\leq \sup_{x \in \Omega} \sup_{a_F \leq y \leq b_F} \frac{1}{h^2} \int_{-\infty}^{+\infty} |\tilde{f}_n(x, y) - F(x, y)| \left\{ \int_{-\infty}^{+\infty} |K(r)L(s)| dr ds \right\}
\]  

(31)

Using Moreira et al. [8] and assumptions (A4) and (A5) we obtain the result.

**Proof of Proposition 3.1.** It is easy to see that using the classic decomposition

\[
\sup_{x \in \Omega} \sup_{a_F \leq y \leq b_F} |\tilde{f}_n(y/x) - f(y/x)|
\leq \frac{1}{\inf_{x \in \Omega} \tilde{f}_n(x)} \left\{ \sup_{x \in \Omega} \sup_{a_F \leq y \leq b_F} |\tilde{f}_n(x, y) - f(x, y)| \right\}
\leq \frac{1}{\inf_{x \in \Omega} \tilde{f}_n(x)} \left\{ \sup_{x \in \Omega} \sup_{a_F \leq y \leq b_F} |\tilde{f}_n(x) - f(x)| \right\} + \sup_{x \in \Omega} \sup_{a_F \leq y \leq b_F} |\tilde{f}_n(x) - f(x)|
\]  

(32)

An application of Lemma 4.1 and Lemma 4.2 and the assumptions (A4)-(A5) allows us to conclude.
Proof of Theorem 3.1.

We have

\[ \sup_{x \in \Omega} |f(\hat{\theta}_n(x)/x) - f(\theta(x)/x)| \]
\[ \leq \sup_{x \in \Omega} |f(\tilde{\theta}_n(x)/x) - f(\tilde{\theta}_n(x)/x)| \]
\[ + \sup_{x \in \Omega} f_n(\tilde{\theta}_n(x)/x) - f(\theta(x)/x) | \]
\[ \leq \sup_{x \in \Omega} \sup_{a \leq y \leq b} f_n(y/x) - f(y/x) | \]
\[ + \sup_{x \in \Omega} \sup_{a \leq y \leq b} f_n(y/x) - \sup_{a \leq y \leq b} f(y/x) | \]
\[ \leq 2 \sup_{x \in \Omega} \sup_{a \leq y \leq b} |f_n(y/x) - f(y/x)| \] (33)

The Taylor expansion of \( f(\cdot/x) \) in the neighborhood of \( \theta(x) \) gives

\[ |f(\hat{\theta}_n(x)/x) - f(\theta(x)/x)| = \frac{1}{2} (\hat{\theta}_n(x) - \theta(x))^2 f^{(2)}(\theta(x)/x) \] (34)

where \( \hat{\theta}(x) \) is between \( \hat{\theta}_n(x) \) and \( \theta(x) \).

Using (A2), we have

\[ \sup_{x \in \Omega} |\hat{\theta}_n(x) - \theta(x)| \leq 2 \sqrt{\frac{\sup_{x \in \Omega} \sup_{a \leq y \leq b} |f_n(y/x) - f(y/x)|}{f^{(2)}(\hat{\theta}(x)/x)}} \] (35)

with Proposition 3.1, the proof is complete.

Proof of Theorem 3.2.

From (20) we have the following decomposition

\[ \sqrt{nh^4} \left( \hat{\theta}_n(x) - \theta(x) \right) = \sqrt{nh^4} \frac{f_n^{(0,1)}(x, \theta(x)) - f_n^{(0,1)}(x, \theta(x))}{f_n^{(0,2)}(x, \tilde{\theta}_n(x))} \]
\[ + \sqrt{nh^4} \frac{f_n^{(0,1)}(x, \theta(x)) - E[f_n^{(0,1)}(x, \theta(x))]}{f_n^{(0,2)}(x, \tilde{\theta}_n(x))} \] (36)
\[
\sqrt{nh^4} \left( \tilde{f}_{n}^{(0,1)}(x, \theta(x)) - \tilde{f}_{n}^{(0,1)}(x, \theta(x)) \right) \\
\leq \sqrt{nh^4} \frac{\sup_{x \in \Omega} \alpha_n}{nh^3} \frac{\alpha}{g_n(x)} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right) \frac{\theta(x) - Y_i}{h}
\]

(37)

Keep in mind that \( \sup_{x \in \Omega} \left| \frac{\alpha_n}{g_n(x)} - \frac{\alpha}{g(x)} \right| = O \left( \frac{1}{\sqrt{n h_n}} \right) \) a.s (see, [7]). Using the usual kernel method, by adding and subtracting the expectation of \((nh^3)^{-1} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right) \left( \frac{\theta(x) - Y_i}{n} \right)\) and conclude that \( J_1 \) is negligible.

J3 has the following definition:

**Lemma 4.3.** Under assumptions (A1), (A4), (A5), (A7), \( J_3 \to 0 \) a.s as \( n \to \infty \).

**Proof** From

\[
\tilde{f}_{n}^{(0,1)}(x, \theta(x)) = \frac{\alpha}{nh^3} \sum_{i=1}^{n} G(Y_i)^{-1} K \left( \frac{x - X_i}{h} \right) \frac{Y - Y_i}{h}
\]

(38)

Then

\[
\sqrt{nh^4} \left( \tilde{f}_{n}^{(0,1)}(x, \theta(x)) - \tilde{f}_{n}^{(0,1)}(x, \theta(x)) \right) \\
= \frac{J_1 + J_2 + J_3}{\tilde{f}_{n}^{(0,2)}(x, \theta_n(x))}
\]

We establish that the numerators of \( J_1 \) and \( J_3 \) are negligible, and that \( J_2 \) is normally distributed. In addition to the probability that the denominator will eventually converge in probability to the value \( f^{(0,2)}(x, \theta(x)) \).

For the first term \( J_1 \), we have
\[ J_3 = E \left[ \frac{\partial^2}{\partial x \partial \theta} \right] = \frac{\alpha}{nh^3} \int_{\mathbb{R}^2} \frac{1}{G(u)} K \left( \frac{x-u}{h} \right) L^{(1)} \left( \frac{\theta(x)-v}{h} \right) f(u,v) \, du \, dv \]

\[ = \sqrt{\frac{nh^3}{n}} \int_{\mathbb{R}^2} K \left( \frac{x-u}{h} \right) H^{(1)} \left( \frac{\theta(x)-v}{h} \right) f(u,v) \, du \, dv \]

\[ = \frac{1}{\sqrt{nh^3}} \int_{\mathbb{R}^2} K \left( \frac{x-u}{h} \right) H^{(1)} \left( \frac{\theta(x)-v}{h} \right) f(u,v) \, du \, dv. \]  

\[ (39) \]

Integration by parts using assumptions (A1) and (A4) and using the Taylor expansion of \( f^{(0,1)}(x-\theta h, \theta(x)-s h_n) \) around \((x, \theta(x))\) to the order of \( h^3 \) we obtain \( J_3 = O(nh^7) \).

**Lemma 4.4.** Under assumptions (A1), (A4) and (A5)

\[ \text{Var}(J_2) \to \alpha^2 f(x, \theta(x)) \int_{\mathbb{R}^2} \frac{K^2(r)}{G^2(r)} \left( L^{(1)}(s) \right)^2 \, dr \, ds \quad \text{as } n \to \infty \]  

\[ (40) \]

**Proof**

\[ \text{Var}(J_2) = \text{Var} \left( \frac{\alpha}{nh^2} \sum_{i=1}^{n} G^{-1}(Y_i) K \left( \frac{x-X_i}{h} \right) L^{(1)} \left( \frac{\theta(x)-Y_i}{h} \right) \right) \]

\[ = \frac{\alpha^2}{nh^2} E \left\{ \frac{1}{G^2(X_1)} K^2 \left( \frac{x-X_1}{h} \right) \left( L^{(1)} \right)^2 \left( \frac{\theta(x)-Y_1}{h} \right) \right\} \]

\[ - \frac{\alpha^2}{nh^2} E \left\{ \frac{1}{G(X_1)} K \left( \frac{x-X_1}{h} \right) L^{(1)} \left( \frac{\theta(x)-Y_1}{h} \right) \right\} \]

\[ = U_{1n} + U_{2n}. \]

Where

\[ U_{1n} = \frac{\alpha^2}{nh^2} \int_{\mathbb{R}^2} K^2(r) \left( L^{(1)} \right)^2 f(x-r h, \theta(x)-s h) \, dr \, ds \]

\[ = \frac{\alpha^2}{nh^2} f(x, \theta(x)) \left( L^{(1)} \right)^2 (s) \, dr \, ds + o(1) \quad \text{as } n \to \infty \]  

\[ (42) \]

Moreover, by Lemma 4.3 : \( U_{2n} = J_3^2 \to 0 \) as \( n \to \infty \).
In one way or another, this leads to a conclusion.
We now focus on the denominator in (4.11). When $\bar{\theta}_n(x)$ converges almost surely to $\theta(x)$; its consistency will be established if we prove lemma 4.5

**Lemme 4.5.** Under assumptions (A4), (A5), (A7) and (A8)

$$\sup_{a_F \leq y \leq b_F} \left| f_n^{(0,2)}(x, y) - f^{(0,2)}(x, y) \right| \rightarrow 0 \text{ p.s. } qd \ n \rightarrow \infty. \quad (43)$$

**Proof** Recall that,

$$\left| f_n^{(0,2)}(x, y) - f^{(0,2)}(x, y) \right|
\leq \left| f_n^{(0,2)}(x, y) - \tilde{f}_n^{(0,2)}(x, y) \right| + \left| \tilde{f}_n^{(0,2)}(x, y) - f^{(0,2)}(x, y) \right| \quad (44)$$

for $\gamma_{1,n}(x, y)$ we have:

$$\sup_{a_F \leq y \leq b_F} \left| f_n^{(0,2)}(x, y) - \tilde{f}_n^{(0,2)}(x, y) \right|
\leq \frac{1}{nh^4} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right) L(2) \left( \frac{y - Y_i}{h} \right) \left| \frac{\alpha_n}{G_n(y)} - \frac{\alpha}{G(y)} \right| \quad (45)$$

As indicated in [7], $\left| \frac{\alpha_n}{G_n(y)} - \frac{\alpha}{G(y)} \right| \rightarrow 0$ as $n \rightarrow \infty$, Under (A4) and (A5) we have $\gamma_{1,n}(x, y) \rightarrow 0$.

For $\gamma_{2,n}(x, y)$:

$$\left| f_n^{(0,2)}(x, y) - f^{(0,2)}(x, y) \right| = \frac{\alpha}{nh^4} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right) L(2) \left( \frac{y - Y_i}{h} \right) - f^{(0,2)}(x, y), \quad (46)$$

by integrating by parts twice with respect to the second component and using a change of variable, it follows that
\[ \gamma_{2,n}(x,y) = \int_{\mathbb{R}^2} K \left( \frac{x-u}{h} \right) L^{(2)} \left( \frac{y-v}{h} \right) f(u,v) du dv - f^{(0,2)}(x,y) \]

\[ = \int_{\mathbb{R}^2} K(r) L(s) \left\{ f^{(0,2)}(x-r, y-s) - f^{(0,2)}(x,y) \right\} dr ds. \] (47)

Using Taylor expansion in the vicinity of \((x,y)\) we obtain

\[ |\gamma_{2,n}(x,y)| \leq |K(r) L(s) \{ r f^{(1,2)}(\bar{x}, \bar{y}) + s f^{(0,3)}(\bar{x}, \bar{y}) \}| dr ds, \] (48)

where \((\bar{x}, \bar{y})\) is between \((x, y)\) and \((x-r, y-s)\), using (A1) and (A8), it follows that \(\gamma_{2,n}(x,y) \to 0\).

The final step in proving Theorem 4.2 is to show the Berry-Esseen condition for \(J_2\). Thus, given (11). Let: \(J_2 = \sum_{i=1}^{n} \Gamma_{i,n}(x,y)\), where

\[ \Gamma_{i,n}(x,y) = \frac{1}{nh^2} \left\{ \bar{f}_{i}^{(0,1)}(x, \theta(x)) - E \left[ \bar{f}_{i}^{(0,1)}(x, \theta(x)) \right] \right\} \]

\[ = \frac{\alpha}{nh^2} \sqrt{nh^4} \left\{ G^{-1}(Y_1) K \left( \frac{x-x_i}{h} \right) H^{(1)} \left( \frac{\theta(x) - Y_1}{h} \right) - E \left[ G^{-1}(Y_1) K \left( \frac{x-x_i}{h} \right) H^{(1)} \left( \frac{\theta(x) - Y_1}{h} \right) \right] \right\} \]

\[ \leq \frac{\alpha}{nh^2} \left\{ G^{-1}(Y_1) K \left( \frac{x-x_i}{h} \right) H^{(1)} \left( \frac{\theta(x) - Y_1}{h} \right) - E \left[ G^{-1}(Y_1) K \left( \frac{x-x_i}{h} \right) H^{(1)} \left( \frac{\theta(x) - Y_1}{h} \right) \right] \right\} \] (49)

Under (A1), (A4), (A5) and (A7) we prove that \(\sum_{i=1}^{n} E \left| \Gamma_{i,n}(x,y) \right|^3 < \infty\).

Applying the \(C_r\) inequality (see Loève [6]), we have,

\[ E \left( \left| \Gamma_{i,n}(x,y) \right|^3 \right) \leq \frac{2^2}{(nh^2)^3} E \left[ \left| \alpha G^{-1}(Y_1) K \left( \frac{x-x_i}{h} \right) H^{(1)} \left( \frac{\theta(x) - Y_1}{h} \right) \right|^3 \right] \]

\[ + \frac{2^2}{(nh^2)^3} E \left\{ E \left[ \alpha G^{-1}(Y_1) K \left( \frac{x-x_i}{h} \right) H^{(1)} \left( \frac{\theta(x) - Y_1}{h} \right) \right] \right\} \] (50)

The two expectation terms in (50) are bounded by virtue of (A1), (A4) and (A7), we have, \(\sum_{i=1}^{n} E \left( \left| \Gamma_{i,n}(x,y) \right|^3 \right) = o(1)\). This completes the proof of Theorem 3.2.

**Proof of Theorem 3.3.** By applying Lemma 2 in [5],
\[ E\left(f_n(y/x)\right) = E\left[ \frac{\alpha_n}{nh} \sum_{i=1}^{n} G_n(Y_i)^{-1} K\left(\frac{x-X_i}{h}\right)L\left(\frac{y-Y_i}{h}\right) \right]. \] (51)

It can be estimated by
\[
\frac{E\left[\alpha_n\sum_{i=1}^{n} G_n(Y_i)^{-1} K\left(\frac{x-X_i}{h}\right)L\left(\frac{y-Y_i}{h}\right) \right]}{E\left[\frac{\alpha_n}{nh} \sum_{i=1}^{n} G_n(Y_i)^{-1} K\left(\frac{x-X_i}{h}\right) \right]}.
\] (52)

The denominator moment consists of
\[
E\left[ \frac{\alpha_n}{nh} \sum_{i=1}^{n} G_n(Y_i)^{-1} K\left(\frac{x-X_i}{h}\right) \right] = \frac{\alpha_n}{n} G_n(Y_i)^{-1} E\left[ K\left(\frac{x-X_i}{h}\right) \right] = \frac{\alpha_n}{n} G_n(Y_i)^{-1} \left[ f(x) + \frac{h^2}{2} \mu_2(K)f''(x) + o(h^2) \right]
\] (53)

and the denominator
\[
E\left[ \frac{\alpha_n}{nh^2} \sum_{i=1}^{n} G_n(Y_i)^{-1} K\left(\frac{x-X_i}{h}\right)L\left(\frac{y-Y_i}{h}\right) \right]
= \frac{\alpha_n}{nh^2} G_n(Y_i)^{-1} K\left(\frac{x-X_i}{h}\right) \left[ f(y/x) + \frac{h^2}{2} \mu_2(L) \frac{\partial^2}{\partial y^2} f(y/x) + O(h^2) \right]
\] (54)

Applying the current formula, a second time, we obtain
\[
\frac{\alpha_n}{nh^2} G_n(Y_i)^{-1} E\left[ K\left(\frac{x-X_i}{h}\right) \left[ f(y/x) + \frac{h^2}{2} \mu_2(L) \frac{\partial^2}{\partial y^2} f(y/x) + O(h^2) \right] \right] = \frac{\alpha_n}{nh^2} G_n(Y_i)^{-1} E\left[ K\left(\frac{x-x_i}{h}\right) Q(x,y) \right].
\] (55)
and

\[
\frac{\alpha}{nh^2} G_n(Y_i)^{-1} E \left[ K \left( \frac{x - X_i}{h} \right) Q(x, y) \right] = \frac{\alpha}{nh^2} G_n(Y_i)^{-1} \times \left[ f(x)Q(x, y) + \frac{h^2}{2} \mu_2(K) \frac{\partial^2}{\partial x^2} (f(x)Q(x, y)) + o(h^4) \right]
\]

\[
= \frac{\alpha}{nh^2} G_n(Y_i)^{-1} f(x) \left[ f(y/x) + \frac{h^2}{2} \mu_2(L) \frac{\partial^2}{\partial y^2} f(y/x) \right] + \frac{h^2}{2} \mu_2(K) f(x) \frac{\partial^2}{\partial x^2} f(y/x) + o(h^4).
\]

(56)

Using this result \( \frac{1}{\delta + s} = \frac{1}{s} + \frac{\delta}{s^2} + o(\delta^2) \). We get

\[
E(\hat{f}_n(y/x)) = f(y/x) + \frac{h^2}{2} \mu_2(K) \frac{\partial^2}{\partial x^2} f(y/x)
\]

\[
+ \frac{h^2}{2} \mu_2(L) \frac{\partial^2}{\partial y^2} f(y/x) + O(h^2).
\]

(57)

Hyndman’s lemma (see, [5]) approximates the variance of \( \hat{f}_n(y/x) \) by

\[
\text{Var}(\hat{f}_n(y/x)) = \text{Var} \left[ \frac{\alpha}{nh^2} \sum_{i=1}^{n} G_n(Y_i)^{-1} K \left( \frac{x - X_i}{h} \right) \right] = \frac{\alpha^2 G_n(Y_i)^{-2} f(x)f(y/x)R(K)R(L)}{n^2 h^3}.
\]

(58)

The numerator yields

\[
\text{Var} \left[ \frac{\alpha}{nh^2} \sum_{i=1}^{n} G_n(Y_i)^{-1} K \left( \frac{x - X_i}{h} \right) \right] = \frac{\alpha^2 G_n(Y_i)^{-2} f(x)f(y/x)R(K)R(L)}{n^2 h^3}.
\]

(59)

Approximating the denominator yields

\[
E^2 \left[ \frac{\alpha}{nh^2} \sum_{i=1}^{n} G_n(Y_i)^{-1} K \left( \frac{x - X_i}{h} \right) \right] = \frac{\alpha^2 G_n(Y_i)^{-2} f^2(x)}{n h^3} + O(h^2).
\]

(60)

Then
\[ \text{Var}(\hat{f}_n(y/x)) = \frac{f(y/x)R(K)R(L)}{nh^3f(x)} + O\left(\frac{1}{nh^2}\right). \] (61)

This completes the proof of Theorem 3.2.

5 CONCLUSION

The study establishes the asymptotic normality and high consistency of the suggested estimator under particular regularity assumptions. With strong mathematical justifications, we demonstrate the effectiveness of their approach in estimating the conditional mode function in complex statistical contexts.

The paper also cites previous statistical studies that advanced our understanding of the development of nonparametric conditional mode function estimate methods. All things considered, this study will be a helpful resource for scholars and practitioners interested in nonparametric estimation methods for doubly truncated data. Researchers might significantly advance the field of statistical methods and applications by furthering the study of nonparametric estimation for conditional mode functions in the setting of doubly truncated data.

The problem when the distribution of the truncated data is assumed to belong to a given parametric family is investigated by [7]. Consequently, in our further research we will study the conditional mode estimation in semi parametric context. Furthermore, this paper does not address the bandwidth selection procedures. This topic is likely to be explored in future studies.
REFERENCES


