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S-asymptotically ω-Periodic Solutions of generalized Liénard Equations

Soluções S-assimptóticas ω-periódicas de equações de Liénard generalizadas

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ABSTRACT
The S-asymptotically ω-periodic functions are a continuous and bounded functions from the real axis to a Banach space that converges to a periodic functions as t tends to infinity. Starting from the zero solution, we prove in this work the existence and uniqueness of S-asymptotically ω-periodic solution of generalized Liénard's differential Equation. We stady after that the regular dependence of this solution with a certain parameter in Banach space, present in our equation, and with the forcing term hwo possesses a similar nature as the later. For this, our approach will be to use a perturbation method around an equilibrium. More precisely when the forcing is a Sasymptotically ω-periodic function we study the differentiable dependence of the S-asymptotically ω-periodic solution of Liénard equation. In this study, we changed our initial objective, which was to search for na S asymptotically ω-periodic solution for our Lienard equation, a problem relating to dynamical systems, towards an approach based on functional analysis. Concretely, we adopted a strategy consisting of using the implicit function theorem on a specific operator that we defined in our workspaces. This approach allowed us to achieve the objective stated in our main theorem. To realize our aim, we use the Nemytskii operators (also called superposition operators) and state some properties on these operators. We have also extended the well-established result on the almost periodic function of the derivative of an almost periodic function to the context of S-asymptotically ω-periodic cases. Finally, and to close our work, we give a corollary which presents a particular case of our main theorem.

Keywords: S-asymptotically ω-periodic functions, Liénard equation, Nemytskii operator, implicit function theorem.

RESUMO
As funções S-assimptoticamente ω-periódicas são funções contínuas e limitadas do eixo real para um espaço de Banach que convergem para funções periódicas à medida que t tende ao infinito. A partir da solução zero, provamos neste trabalho a existência e a singularidade da solução S-assintótica ω-periódica da equação diferencial de Liénard generalizada. Posteriormente, verificamos que a dependência regular dessa solução com um determinado parâmetro no espaço de Banach, presente em nossa equação, e com o termo forçante hwo possui uma natureza semelhante à última. Para isso, nossa abordagem será usar um método de perturbação em torno de um equilíbrio. Mais precisamente, quando a forçante é uma função sassimptoticamente ω-periódica, estudamos a dependência diferenciável da solução S-assimptoticamente ω-periódica da equação de Liénard. Neste estudo, mudamos nosso objetivo inicial, que era procurar uma solução S assintoticamente ω-periódica para nossa equação de Liénard, um problema relacionado a sistemas dinâmicos, para uma abordagem baseada em análise funcional. Em termos concretos, adotamos uma estratégia que consiste em usar o teorema da função implícita em um operador específico que definimos em nossos espaços de trabalho. Essa abordagem nos permitiu atingir o objetivo declarado em nosso teorema principal. Para atingir nosso objetivo, usamos os operadores Nemytskii (também chamados de operadores de superposição) e definimos algumas propriedades sobre esses operadores. Também estendemos o resultado bem estabelecido sobre a função quase periódica da derivada de uma função quase periódica para o contexto de casos S-assintóticamente ω-periódicos. Por fim, para encerrar nosso trabalho, fornecemos um corolário que apresenta um caso particular de nosso teorema principal.

Palavras-chave: funções S-assintóticamente ω-periódicas, equação de Liénard, operador de Nemytskii, teorema da função implícita.

1 INTRODUCTION

Recently, a new class of functions generalizing that of periodic functions has been defined, this is the class of S-asymptotically ω - periodic functions. This class of functions also generalized that of asymptotically ω - periodic functions. In [12] Lizama and N'Guèrèkata show a relation between S-asymptotically $\omega -$ periodic functions and several subspaces of $BC(\mathbb{R},X)$, where X is a Banach space.

The existance and uniqueness of an S-asymptotically $\omega -$ periodic solution has great importance in the qualitative study of the theory of differential equations due to has their applications in many fields such as mathematical biology, physics, control theory and other fields. Among these application we find as an important oscillatory model in the field of physics, the forced Liénard's equation.

The leading work investigation for the existence of periodic solution of the equation

$$x''(t) + \phi\big(x(t), x'(t)\big) \cdot x'(t) + \psi\big(x(t)\big) = 0, \quad (1)$$

was established by Levinson and Smith [11].

The equation

$$x''(t) + f(x(t), x'(t), q) \cdot x'(t) + g(x(t), q) = b(t), \quad (2)$$

is the generalized Liénard equation and is based on a more realistic modelling. In [6], the authors prove the existence of periodic solution of the perturbed generalized Liénard equation, such that (2) has at last one periodic solution.

Results on the differentiable dependence were established in [1], [3] and in [5] for Several kinds of solutions of Liénard equations. In [4], the authors prove the existence of S-asymptotically ω-periodic solutions of

$$x'(t) = A(t)x(t) + f\big(t, x(t), u(t)\big), x(0) = \xi. \quad (3)$$

More precisely when u is a S-asymptotically ω-periodic function, the differentiable dependence of the S-asymptotically ω-periodic solution of (3) with respect to u and the initial value ξ.

The aim of this paper is to present some new results concerning the existence and uniqueness of S-asymptotically Periodic solution, starting from the zero solution when $q = 0$, of the equation in the form (2) where q is a parameter in a Banach space Q, b is S-asymptotically Periodic function and $f: \mathbb{R}^2 \times Q \to \mathbb{R}$, $g: \mathbb{R} \times Q \to \mathbb{R}$ two functions.

The paper is organized as follows: in the Section 2, we precise the notations of the function spaces and we give the important lemma which are used in the paper. In the Section 3, we state the main results of the equation (2) and we give the proof of the main theorem. We establish a corollary from the equation

$$x''(t) + \phi\big(x(t), x'(t)\big) \cdot x'(t) + \psi\big(x(t)\big) = e(t). \quad (5)$$

Where:

e is S-asymptotically ω-periodic function and $\phi: \mathbb{R}^2 \to \mathbb{R}, \psi: \mathbb{R} \to \mathbb{R}$ two functions.

The goal of this work is to finding S-asymptotically ω-periodic solution for Lienard's equations. This type of equations often describe systems with nonlinear dynamics, including oscillations, limit cycles, and chaos. This solution provide insights into the behavior of the system and how it evolves over time. This helps in understanding whether the system converges to periodic motion, approaches a stable equilibrium, or exhibits other asymptotic behavior.

In engineering, physics, and biology, understanding the long-term behavior of dynamical systems is essential for designing and controlling systems. Knowing whether a system settles into periodic behavior or approaches a stable state informs decisions about system design, control strategies, and predicting system responses.

2 NOTATION

X and Y are Banach spaces. $BC^0(\mathbb{R}, X)$ denotes the space of the bounded continuous functions from $\mathbb{R}$ into X. $\| x \|_\infty := \sup_{t\in\mathbb{R}} |x(t)|$ is the usual norm on $BC^0(\mathbb{R}, X)$. For $k = \{1,2\}$, let

$$BC^k(\mathbb{R}, X) = \{u \in C^k(\mathbb{R}, X); \forall j = 1, \dots k u^j \in BC^0(\mathbb{R}, X)\}. \quad (6)$$

Endowed with the norm $\| u \|_{BC^k} := \| u \|_\infty + \sum_{j=1}^{j=k} \|u^j\|_\infty$, $BC^k(\mathbb{R}, X)$ is a Banach space.

Definition 2.1. [8,2] Let $\omega \in (0, \infty)$. A function $u \in BC^0(\mathbb{R}, X)$ is called S-asymptotically ω-periodic when it satisfies the following condition:

$$\lim_{|t|\to\infty} (u(t+\omega) - u(t)) = 0.$$

The space of such functions is denoted by $SAP_\omega^0(X)$.

$(SAP_\omega^0(X), \|\cdot\|_\infty)$ is a Banach space (Theorem 3.3 in [2]). For $k = \{1,2\}$, let

$SAP_\omega^k(X) = \{u \in SAP_\omega^0(X); \forall j = 1, k\ u^j \in SAP_\omega^0(X)\}$.

Endowed with the norm $\| u \|_{BC^k}$, $SAP_\omega^k(X)$ is a Banach space.

$\mathcal{P}_b(X)$ denotes the set of the bounded subsets of X.

Definition 2.2. [4]. Let $\phi: X \to Y$ be a mapping. We say that ϕ is boundedly uniformly continuous when the following conditions are fulfilled.

a) For all $B \in \mathcal{P}_b(X), \phi(B) \in \mathcal{P}_b(Y)$.

b) For all $B \in \mathcal{P}_b(X)$, the restriction $\phi_{|_B}$ is uniformly continuous.

We denote by $U_b(X, Y)$ the set of such mappings.

After these conditions on the continuity, we consider notions which concern the (Fréchet) differentiability.

Definition 2.3. [4]. Let $\phi: X \to Y$ be a mapping. We say that ϕ is uniformly C^1 on the bounded subsets of X when

a) $\phi \in U_b(X, Y)$.

b) $\phi \in C^1(X, Y)$.

c) $D\phi \in U_b(X, \mathcal{L}(X, Y))$.

We denote by $U_b^1(X, Y)$ the space of such mappings.

Lemma 2.4. [4]. Let $\phi \in U_b^1(X, Y)$. Then the operator of Nemytskii N_ϕ defined by

$$N_\phi(u) := [t \mapsto \phi(u(t))],$$

belongs to $C^1(SAP_\omega^0(X), SAP_\omega^0(Y))$, and for all $u, h \in SAP_\omega^0(X)$, we have

$$DN_\phi(u)h = [t \mapsto D\phi(u(t))h(t)].$$

Definition 2.5. The family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is said to be a C_0-semigroup if

a) $T(0) = I_{\mathcal{L}(X)}$, the identity operator

b) $\forall (s,t) \geq 0,\ T(s+t) = T(s) \circ T(t)$

c) $\forall x \in X,\ \lim_{t\to 0^+} T(t)x = x$.

The operator $A: D(A) \subset X \longrightarrow X$ is called the infinitesimal generator of the C_0-semigroup T if

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}, \text{ where } D(A) = \left\{ x \in X, \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ existe} \right\}.$$

Definition 2.6. We say that a C_0-semigroup, $(T(t))_{t \geq 0}$, is exponentially stable if there exist $K > 0, w < 0$ such that

$\| T(t) \| \leq Ke^{wt}$, for all $t \geq 0$.

For a bounded linear operator A we have

$$T(t) = e^{tA} = \sum_{0}^{\infty} \frac{t^n A^n}{n!}.$$

Definition 2.7. ([13], p. 126). A function $x \in C^0(\mathbb{R}_+, X)$ is called a classical solution of the problem

$$u'(t) = Au(t) + B(t). \qquad (7)$$

when $x \in C^0([0, \infty), X) \cap C^1((0, \infty), X), x(t) \in \mathcal{D}(A(t))$ for all $t \in \mathbb{R}_+$and it satisfies (2.1).

Definition 2.8. [10, p.11] A function $x \in C^0(\mathbb{R}, \mathbb{E})$ is said to be mild solution of (7) if x satisfies the equation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)f(s)ds,$$

for any $a \in \mathbb{R}$ and any $t \geq a$, and where A is the infinitesimal generator of the C_0-semigroup T.

Lemma 2.9. Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that all its eigenvalues have a real part different from zero. Then, for all $\in SAP_\omega^0(\mathbb{R}^n)$), there exists a unique S asymptotically ω-periodic solution of the differential equation (7).

Proof. We denote by $\lambda_1, \ldots, \lambda_k$ the different eigenvalues of $A, k \leq n$. We fix $B \in SAP_\omega^0(\mathbb{R}^n)$

The case where the real part of $\lambda_j, \Re\lambda_j < 0$ for all $j = 1, \ldots, k$.

Since in this case the origine is a sink for the dynamical system $u' = Au$ and by using Theorem 1 in [9] (p. 145) we know that there exist $K \in (0, \infty)$ and $\omega \in (0, \infty)$ such that

$$\|e^{tA}\|_{\mathcal{L}} \leq K.e^{-t\omega} \forall t \in [0, \infty),$$

then $(e^{tA})_{t\geq 0}$ is an exponentially stable C_0-semigroup.

Note that to say that u is a classical solution of (7) is equivalent to say that that u is a mild solution of (7) since the domain of A is $\mathbb{R}^n$. And then, Corollary 3.6 in [12] ensures that there exists a unique S-asymptotically ω-periodic solution of (7).

The case where $\Re\lambda_j > 0$ for all $j = 1, \ldots, k$.

Note that u is a solution of (7) on $\mathbb{R}$ if and only if v, defined by $v(t) := u(-t)$, is a solution on $\mathbb{R}$ of the equation

$$v'(t) = -Av(t) - B(-t). \quad (8)$$

Since the eigenvalues of $-A$ are $-\lambda_1, \ldots, -\lambda_k$, the dynamical system (8) enters in the setting of the first step.

For $\omega > 0$ and by change of variable $t = -\omega - s$, we obtained

$$\lim_{|t|\to\infty} (-B(-t-\omega) + B(-t)) = \lim_{|s|\to\infty} (B(s+\omega) + B(s)) = 0,$$

then $t \mapsto -B(-t)$ belongs to $SAP_\omega^0(\mathbb{R}^n))$, by using the first step, we know that there exists a unique S-asymptotically ω-periodic solution, denoted by v, of (8).

Consequently the function u, defined by $u(t) := v(-t)$, is the unique S-asymptotically ω-periodic solution of (7).

For the setting of the complex case associated to (7).

$$z'(t) = Az(t) + \rho(t),\ z(t) \in \mathbb{C}^n, \rho(t) \in \mathbb{C}^n. \quad (9)$$

Let $z(t) = x(t) + i.y(t)$ with $x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^n$, be a solution of (9). Since A is real, x is a solution of (7) with $B = \Re\rho$, and y is a solution of (7) with $B = \Im\rho$.

If we assume that $\Re\lambda_j < 0$ for all $j = 1, \dots, k$, we can extend the first case to the complex equation (9). When $\rho \in SAP_\omega^0(\mathbb{C}^n)$, then $\Re\rho$ and $\Im\rho$ belong to $SAP_\omega^0(\mathbb{R}^n)$, and by using the first case, there exists a unique S asymptotically ω-periodic solution x of (7) with $B = \Re\rho$ and there exists a unique S-asymptotically ω-periodic solution y of (7) with $B = \Im\rho$. Therefore $z := x + i.y$ is the unique S-asymptotically ω-periodic solution of (9).

Following the same reasoning, we can extend the case where $\Re\lambda_j > 0$ for all $j = 1, \dots, k$. And so, we obtain that, under this last condition, for all $\rho \in SAP_\omega^0(\mathbb{C}^n)$ there exists a unique S-asymptotically ω-periodic solution of (9).

The case where there exists an integer number m such $1 \leq m < \nu$ satisfying $\Re\lambda_j < 0$ when $1 \leq j \leq m$ and $\Re\lambda_j > 0$ when $m < j \leq k$.

Denoting by X_j the generalized eigenspace associated to λ_j as defined in [9] p. 110 , we set

$$X_- := X_1 \oplus \dots \oplus X_m \text{ and } X_+ := X_{m+1} \oplus \dots \oplus X_k.$$

We have $\mathbb{C}^n = X_- \oplus X_+$with $A(X_-) \subset X_-$and $A(X_+) \subset X_+$. We denote by $A_-$and $A_+$the linear restrictions of A to $X_-$and X_+respectively. The eigenvalues of A_-have a real part which is negative and the eigenvalues of A_+have a real part which is positive. When $B \in SAP_\omega^0(\mathbb{R}^n)$ then we can consider $B = B + i.0$ as an element of $SAP_\omega^0(\mathbb{C}^n)$. We set $B = B_- \oplus B_+$where $B_- \in SAP_\omega^0(X_-)$and $B_+ \in SAP_\omega^0(X_+)$. By using the third case, we know that there exists

$$z_- \in SAP_\omega^1(X_-) \text{and } z_+ \in SAP_\omega^1(X_+),$$

which is the unique S-asymptotically ω-periodic solution of

$$z_-'(t) = A_-.z_-(t) + B_-(t) \text{ and } z_+'(t) = A_+.z_+(t) + B_+(t),$$

respectively. Then $z := z_- \oplus z_+$is the unique S-asymptotically ω-periodic solution of (9) with $\rho = B$, and consequently $x := \Re z$ is the unique S-asymptotically ω-periodic solution of (7).

3 THE MAIN RESULT

Theorem 3.1. Under the following list of conditions

(H1) $f \in U_b^1(\mathbb{R}^2 \times Q, \mathbb{R}), g \in U_b^1(\mathbb{R} \times Q, \mathbb{R})$

(H2) $g(0,0) = 0$

(H3) $f(0,0,0) \neq 0$ when $f(0,0,0)^2 < 4\frac{\partial g(0,0)}{\partial x}$, and

$\frac{\partial g(0,0)}{\partial x} \neq 0$ when $f(0,0,0)^2 \geq 4\frac{\partial g(0,0)}{\partial x}$,

there exist a neighborhood U of 0 in $SAP_\omega^0(\mathbb{R})$, a neighborhood V of 0 in $SAP_\omega^2(\mathbb{R})$ and a neighborhood W of 0 in Q and a C^1-mapping x^*, from $U \times W$ into V which satisfies the following conditions.

- $x^*(0,0) = 0.$
- For all $b \in U$ and for all $q \in W, x^*(b, q)$ is an S-asymptotically ω-periodic solution of (2).
- If $x \in V$ is an S-asymptotically ω-periodic solution of (2) with $b \in U$ and $q \in W$, then we have $x = x^*(b, q)$.

Proof. To prouve our main result, we use the implicit function theorem ([7], p. 61) on $\Gamma(x, b, q) = 0$, in neighborhood of $(0,0,0)$, where Γ is an operator form $SAP_\omega^2(\mathbb{R}) \times SAP_\omega^0(\mathbb{R}) \times P$ to $SAP_\omega^0(\mathbb{R})$ by setting

$$\Gamma(x, b, q) := [t \mapsto x''(t) + f(x(t), x'(t), q) \cdot x'(t) + g(x(t), q) - b(t)]. \quad (10)$$

We see that, $x \in SAP_\omega^2(\mathbb{R})$ satisfies $\Gamma(x, b, q) = 0$ if and only if x is an Sasymptotically ω-periodic solution of (2);

Since the following equality holds:

$$\Gamma = \frac{d^2}{dt^2} \circ \pi_1 + N_P \circ \left(R_f \circ (\Delta \circ \pi_1, \pi_3), \frac{d}{dt} \circ (\mathrm{in}_1 \circ \pi_1) \right) + R_g \circ (n_2 \circ \pi_1, \pi_3) - \pi_2, \qquad (11)$$

then Γ is well-defined, where:

- $\frac{d^2}{dt^2}: SAP_\omega^2(\mathbb{R}) \to SAP_\omega^0(\mathbb{R})$ is defined by

$$\frac{d^2}{dt^2} x \coloneqq x''.$$

- $\frac{d}{dt}: SAP_\omega^1(\mathbb{R}) \to SAP_\omega^0(\mathbb{R})$ is defined by

$$\frac{d}{dt} x \coloneqq x'.$$

- in $_1: SAP_\omega^2(\mathbb{R}) \to SAP_\omega^1(\mathbb{R})$ is defined by

$$\mathrm{in}_1(x) := x.$$

- $\mathrm{in}_2: SAP_\omega^2(\mathbb{R}) \to SAP_\omega^0(\mathbb{R})$ is defined by

$$\mathrm{in}_2(x) \coloneqq x.$$

- $\pi_1: SAP_\omega^2(\mathbb{R}) \times SAP_\omega^0(\mathbb{R}) \times Q \to SAP_\omega^2(\mathbb{R})$ defined by

$$\pi_1(x, b, q) := x.$$

- $\pi_2: SAP_\omega^2(\mathbb{R}) \times SAP_\omega^0(\mathbb{R}) \times Q \to SAP_\omega^0(\mathbb{R})$ defined by

$$\pi_2(x, b, q) := b.$$

- $\pi_3: SAP^2_\omega(\mathbb{R}) \times SAP^0_\omega(\mathbb{R}) \times Q \to Q$ defined by

$$\pi_2(x, b, q) \coloneqq q.$$

- $\Delta: SAP^2_\omega(\mathbb{R}) \to SAP^0_\omega(\mathbb{R}) \times SAP^0_\omega(\mathbb{R})$ defined by

$$\Delta(x) \coloneqq (x, x').$$

- $P: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $P(r, s) := r.s$. We consider the Nemystkii operator build on P,

$$N_P: SAP^0_\omega(\mathbb{R}) \times SAP^0_\omega(\mathbb{R}) \to SAP^0_\omega(\mathbb{R}),$$

defined by

$$N_P(u, v) \coloneqq [t \mapsto u(t) \cdot v(t) = P(u(t), v(t))].$$

- If we assimilate a point $q \in Q$ to the constant function $t \mapsto q$ which belongs to $SAP^0_\omega(Q)$, that permits us to look at Q as a closed vector subspace of $SAP^0_\omega(Q)$, then we can consider the following restrictions of the operators N_f and N_g:

$$R_f: SAP^0_\omega(\mathbb{R}^2) \times Q \to SAP^0_\omega(\mathbb{R}),$$

defined by

$$R_f(x, x', q) := [t \mapsto f(x(t), x'(t), q)],$$

and

$$R_g: SAP^0_\omega(\mathbb{R}) \times Q \to SAP^0_\omega(\mathbb{R}),$$

defined by

$$R_g(x,q) := [t \mapsto g(x(t),q)],$$

where N_f is an operator of Nemytskii from $SAP^0_\omega(\mathbb{R}^2) \times SAP^0_\omega(Q) = SAP^0_\omega(\mathbb{R}^2 \times Q)$ into $SAP^0_\omega(\mathbb{R})$. N_g is an operator of Nemytskii from $SAP^0_\omega(\mathbb{R}) \times SAP^0_\omega(Q) = SAP^0_\omega(\mathbb{R} \times Q)$ into $SAP^0_\omega(\mathbb{R})$.

Therefore we must ensure the conditions of implicit functions theorem on $\Gamma(x,b,q) = 0$, in neighborhood of $(0,0,0)$, i.e.:

(C1) $\Gamma(0,0,0) = 0$

(C2) Γ is of class C^1

(C3) $D_x\Gamma(0,0,0)$ is a bijection from $SAP^2_\omega(\mathbb{R})$ onto $SAP^0_\omega(\mathbb{R})$.

3.1 CONDITION (C1)

Under (H2), note that 0 is an S-asymptotically ω-periodic solution of (2), and so following equality holds

$$\Gamma(0,0,0) = 0.$$

3.2 CONDITION (C2)

Since for all $x \in SAP^2_\omega(\mathbb{R})$,

$$\left\| \frac{d^2}{dt^2} x \right\|_\infty \leq \| x \|_{BC^2}, \quad \left\| \frac{d}{dt} x \right\|_\infty \leq \| x \|_{BC^1},$$
$$\|\mathrm{in}_1(x)\|_{BC^1} \leq \| x \|_{BC^2}, \ \|\mathrm{in}_2(x)\|_{BC^0} \leq \| x \|_{BC^2},$$

then the linear operators $\frac{d^2}{dt^2}, \frac{d}{dt}$ and $\mathrm{in}_1, \mathrm{in}_2$ are continuous and consequently they are of class C^1.

Since

$$\|\pi_1(x,b,q)\|_{BC^2} \leq \| (x,b,q) \|_{BC^2 \times BC^0 \times Q},$$
$$\|\pi_2(x,b,q)\|_{BC^0} \leq \| (x,b,q) \|_{BC^2 \times BC^0 \times Q},$$
$$\|\pi_3(x,b,q)\|_Q \leq \| (x,b,q) \|_{BC^2 \times BC^0 \times Q},$$
$$\| \Delta(x) \|_{BC^2 \times BC^1} \leq \| (x) \|_{BC^2},$$

then the linear operators π_1 and π_2, π_3, Δ are continuous and consequently they are of class C^1.

By using Lemma 2.4 and (H1) we know that N_f and N_g are of class C^1 for all $x, y \in SAP_\omega^0(\mathbb{R})$, we have

$$D_x N_f(x, x', q) \cdot y = \left[t \mapsto \frac{\partial f\big(x(t), x'(t), q(t)\big)}{\partial x} \cdot y(t) \right],$$
$$D_x N_g(x, q) \cdot y = \left[t \mapsto \frac{\partial g\big(x(t), q(t)\big)}{\partial x} \cdot y(t) \right].$$

Since the restriction of a C^1-mapping on a Banach subspace is also a C^1 mapping, then R_f and R_g are of class C^1 and, for all $x, y \in SAP_\omega^0(\mathbb{R})$, we have

$$D_x R_f(x, x', q) \cdot y = \left[t \mapsto \frac{\partial f\big(x(t), x'(t), q\big)}{\partial x} \cdot y(t) \right] \tag{12}$$

$$D_x R_g(x, q) \cdot y = \left[t \mapsto \frac{\partial g\big(x(t), q\big)}{\partial x} \cdot y(t) \right]. \tag{13}$$

Since P is a bilinear continuous function, it is of class C^1, and by using Lemma 2.4 we know that N_P is of class C^1.

Since Γ is a composition of operators of class C^1, then it is of class C^1, thus the condition $(\mathrm{C}2)$ is obtained.

3.3 CONDITION (C3)

For all $y \in SAP_\omega^0(\mathbb{R})$, by using the classical formulas of the differential calculus in Banach spaces and (12), (13), we obtain

$$D_x \Gamma(0,0,0) \cdot y = \frac{d^2}{dt^2} y + N_P\big(D_x R_f(0,0,0) \cdot y, 0\big) + N_P\left(R_f(0,0,0), \frac{d}{dt}(y) + D_x R_g(0,0) \cdot y + 0,\right.$$

that implies, for all $t \in \mathbb{R}$,

$$(D_x \Gamma(0,0,0) \cdot y)(t) = y''(t) + f(0,0,0) \cdot y'(t) + \frac{\partial g(0,0)}{\partial x} \cdot y(t),$$

Let $e \in SAP_\omega^0(\mathbb{R})$ and $y \in SAP_\omega^2(\mathbb{R})$. To say that $D_x \Gamma(0,0,0) \cdot y = e$ is equivalent to say that y is an S-asymptotically ω-periodic solution of

$$y''(t) + f(0,0,0) \cdot y'(t) + \frac{\partial g(0,0)}{\partial x} \cdot y(t) = e(t). \qquad (14)$$

which is equivalent to say that $X(t) := \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ is an S-asymptotically ω-periodic solution of

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}'(t) = \begin{bmatrix} 0 & 1 \\ -\frac{\partial g(0,0)}{\partial x} & -f(0,0,0) \end{bmatrix} \cdot \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}(t) + \begin{bmatrix} 0 \\ e(t) \end{bmatrix}. \qquad (15)$$

By the condition (H3), all the eigenvalues of matix $\begin{bmatrix} 0 & 1 \\ -\frac{\partial g(0,0)}{\partial x} & -f(0,0,0) \end{bmatrix}$, denoted by λ_1 and λ_2, have a real part different from zero. In fact, When $\lambda_1, \lambda_2 \in \mathbb{R}$, we have $f(0,0,0)^2 \geq 4\frac{\partial g(0,0)}{\partial x}$ et $\frac{\partial g(0,0)}{\partial x} = \lambda_1 \cdot \lambda_2$. Then, by (H3), we have $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

When $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, we have $f(0,0,0)^2 < 4\frac{\partial g(0,0)}{\partial x}$ and $f(0,0,0) = -2\Re\lambda_1 = -2\Re\lambda_2$, condition (H3) implies that $\Re\lambda_1 \neq 0$ and $\Re\lambda_2 \neq 0$.

Then by using Lemma 2.9 we obtain the existence of a S-asymptotically ω periodic solution X of (15), and then the existence of a unique S-asymptotically ω-periodic solution y of (14). This is the unique $y \in SAP_\omega^2(\mathbb{R})$ which satisfies $D_x\Gamma(0,0,0).y = b$. Then $D_x\Gamma(0,0,0)$ is a bijection from $SAP_\omega^2(\mathbb{R})$ onto $SAP_\omega^0(\mathbb{R})$ and the condition $(C3)$ is obtained.

By using conditions (C1), (C2) and (C3), we can use the implicit function theorem ([7], p. 61), then there exist a neighborhood U of 0 in $SAP_\omega^0(\mathbb{R})$, a neighborhood V of 0 in $SAP_\omega^2(\mathbb{R})$ and a neighborhood W of 0 in Q and a C^1-mapping x^*, from $U \times W$ into V which satisfies the following conditions.

1. $x^*(0,0) = 0$., that the first condition of Theorem 3.1.
2. $\Gamma(x^*(b,q), b, p) = 0$ for all $(b,q) \in U \times W$, that ensures that $x^*(b,q)$ Sasymptotically ω-periodic solution of (1.2) with $b \in U$ and $q \in W$, that is the second conclusion of Theorem 3.1.
3. $\{(x,b,q) \in V \times U \times W : \Gamma(x,b,q) = 0\} = \{(x^*(b,q), b, q) : (b,q) \in U \times W\}$ that implies the last conclusion of Theorem 3.1.

Corollary 3.2. Under the following list of conditions

(H4) $\phi \in U_b^1(\mathbb{R}^2, \mathbb{R}), \psi \in U_b^1(\mathbb{R}, \mathbb{R})$

$(H5)\psi(0) = 0$

(H6) $\phi(0,0) \neq 0$ when $\phi(0,0)^2 < 4\psi_1'(0)$, and $\psi'(0) \neq 0$ when $\phi(0,0)^2 \geq 4\psi'(0)$.,

there exist a neighborhood U^* of 0 in $SAP_\omega^0(\mathbb{R})$, a neighborhood V^* of 0 in $SAP_\omega^2(\mathbb{R})$ and a C^1-mapping x^{**}, from U^* into V^* which satisfies the following conditions.

- $x^{**}(0) = 0.$
- For all $e \in U^*, x^{**}(e)$ is an S-asymptotically ω-periodic solution of (1.4).
- If $x \in V^*$ is an S-asymptotically ω-periodic solution of (1.4) with $e \in U^*$, then we have $x = x^{**}(e)$.

Proof. We set $Q:= SAP_\omega^0(\mathbb{R})$. We define $f: \mathbb{R}^2 \times SAP_\omega^0(\mathbb{R}) \to \mathbb{R}$ by setting $f(x, x', e):= \phi(x)$, and we define $g: \mathbb{R} \times SAP_\omega^0(\mathbb{R}) \to \mathbb{R}$ by setting $g(x, e):= \psi(x)$.

Assumption (H4) implies (H1), assumption (H5) implies (H2), and assumption (H6) implies (A3).

And so we obtain the conclusion by using Theorem 3.1.

4 CONCLUSION

The study aimed to establish the existence of an S-asymptotically ω-periodic solution for a lienard's equation and to explore its dependence on the forcing and a certain parameter present in our equation. In fact, by reformulating the problem into a framework of functional analysis and employing the implicit function theorem on an operator defined by this equation, we have exploiting the regularity of the superposition operator within the defined spaces. This type of solution ensures a certain form of system stability.

Ultimately we hope to improve the quality of our results. In particular, we must prove the global existence of a fairly regular solution in the usual sense instead of a mild solution and finally answer the question of the regularity of the superposition operator for a broader class of forcing.

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