Studies in Engineering and Exact Sciences
ISSN: 2764-0981

Existance of solutions for some class of system quasilinear differentials with nonlocal boundary conditions and with nonlinearity depandant on the first derivate in time scales

Existência de soluções para algumas classes de sistemas diferenciais quasilineares com condições de contorno não locais e com não linearidade dependente da primeira derivada em escalas de tempo

DOI: 10.54021/seesv5n1-084

Recebimento dos originais: 08/04/2024
Aceitação para publicação: 26/04/2024

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ABSTRACT

This work is conserned with the construction of minimal and maximal solution for some equation quasilinear differentials with nonlocal boundary condition with nonlinearity is continuous function dependent on the first derivate in the time scale. In a previous study, see[11] it was found that there exists the extrimals solutions of a quasilinear elliptic system with intergral boundary conditions in the continuous case, and from here we asked the following question using the upper and lower solutions method coupled with monotone iterative technique and we asked the following question: do we have the existence of extrimals solutions for certain classes of systems of differential equations with nonlocal boundary conditions in time scales by same method. We can classify the problem following the monotonicity of the two functions f and g in the true types.Type 1: Increasing qasimonotonic systems.Type 2: Decreasing qasimonotonic systems. Type 3: Mexed qasimonotonic systems. In this paper we show the existence minimuls-maximuls solutions if the problem is of the type 1. If the problem is of the type 2 , we show the existence of maximuls-minimuls solutions. If the problem is of the type 3 , we show existence of least quasilear-solution. In the end, we gave examples of the results presented in this work to prove their validity through two cases: the continuous case, represented by the set of real numbers, and the discontinuous case, represented by the set of integers.

Keywords: Quasilier equation, nonlocal integral boundary condition, upper and lower solutions iterative technique, time scales, p-Laplacian.

RESUMO

Este trabalho trata da construção de soluções mínimas e máximas para algumas equações diferenciais quasilineares com condições de contorno não locais cuja não linearidade é uma função contínua dependente da primeira derivada na escala de tempo. Em um estudo anterior, veja [11], foi constatado que existem soluções extremas de um sistema elíptico quasilinear com condições de contorno intergral no caso contínuo e, a partir daí, fizemos a seguinte pergunta usando o método de soluções superior e inferior acoplado à técnica iterativa monótona e fizemos a seguinte pergunta: existe a existência de soluções extremas para certas classes de sistemas de equações diferenciais com condições de contorno não locais em escalas de tempo pelo mesmo método? Podemos classificar o problema de acordo com a monotonicidade das duas funções f e g nos tipos verdadeiros: Tipo 1: Sistemas qasimonotônicos crescentes; Tipo 2: Sistemas qasimonotônicos decrescentes. Tipo 3: sistemas quasimonotônicos mexidos. Neste artigo, mostramos a existência de soluções mínimas-máximas se o problema for do tipo 1. Se o problema for do tipo 2, mostraremos a existência de soluções maximuls-minimuls. Se o problema for do tipo 3, mostraremos a existência de soluções quasileares mínimas. No final, demos exemplos dos resultados apresentados neste trabalho para provar sua validade em dois casos: o caso contínuo, representado pelo conjunto de números reais, e o caso descontínuo, representado pelo conjunto de números inteiros.

Palavras-chave: equação de Quasilier, condição de contorno integral não local, técnica iterativa de soluções superior e inferior, escalas de tempo, p-Laplaciano.

1 INTRODUCTION

In this work we are interested in the existence of solutions for the following problem

$$\begin{cases} -\left(\varphi_p(u^\Delta)\right)^\Delta = f(t,u^\sigma,v^\sigma), t \in [a,b]_T, \\ -\left(\varphi_q(u^\Delta)\right)^\Delta = g(t,u^\sigma,v^\sigma), t \in [a,b]_T, \\ c_1u(a) - c_2u^\Delta(a) = L_1(u), \\ c_3u(\sigma^2(b)) + c_4u^\Delta(\sigma^2(b)) = L_2(u), \\ d_1v(a) - d_2v^\Delta(a) = L_3(u), \\ d_3v(\sigma^2(b)) + d_4v^\Delta(\sigma^2(b)) = L_4(u), \end{cases} \quad (1)$$

Where:

$\varphi_p(u) = |u|^{p-2} \cdot u$, such that $p > 1$ and $q > 1, c_i, d_i \in \mathbb{R}_+$, for all $i = 0,1,3,4, f: [a,b]_T \times$

$\mathbb{R}^2 \longmapsto \mathbb{R}$ and $g: [a,b]_T \times \mathbb{R}^2 \longmapsto \mathbb{R}$ are continuous functions and $L_i: C([a,\sigma^2(b)]_T) \to \mathbb{R}$ continuous and inreasing functions for all $i = 0,1,3,4$.

The stady of the problems in the times scales with nonlinear boundary conditions have been by several authors using Leray-Shauder fixed points, the upper and lower solutions method, the coincidence degree theory of Mawhin and fixed point theorems in the cones (see [3],[8],[14] and [23]).

This work is devised into three section, we give somes definition and notations in the first section. In the second section we state and show the main result of this paper. Finally the third section we give somes applications to illustrate ours results.

This work is a generalization of the continuous cas to the time scale (see [9]). The defition of upper and lower solutions and the construction depends on the monotonicity of the two functions f and g, then according to C.V. Pao (see [22]), we can classify the systhem (1) following the monotonicity of the two functions f and g in the true types:

Type 1: Increasing qasimonotonic systems.

Type 2: Decreasing qasimonotonic systems.

Type 3: Mixed qasimonotonic systems.

In this paper we show the existence minimuls-maximuls solutions if the system (1) is of the type 1. If the system (1) is of the type 2 , we show the existence of maximuls-minimuls solutions. If the system is of the type 3 , we show existence of least quasilear-solution.

2 DEFINITIONS AND NOTATIONS

In this section, we give some definitions and notations.

Definition 1: *A time scale T is an arbitrary nonempty closed subset of the real numbers.*

Definition 2: *We define the interval $[a,b]_T$ in T by*

$$[a,b]_T := \{t \in T: [a,b] \cap T\},$$

Where:

$a, b \in T$.

Definition 3: *Let* T *be a time scale. The forward jump operator* σ *and the backward jump operator* ρ *are defined by*

$$\sigma(t) := inf\{s > t : s \in T\},$$

And

$$\rho(t) := sup\{s > t : s \in T\}.$$

Definition 4: *Let* T *be a time scale. If* $\sigma(t) > t$, *we say* t *is right-scattered, while if* $\rho(t) < t$, *we say* t *is left-scattered. If* $\sigma(t) = t$, *we say* t *is right-dense, while if* $\rho(t) = t$, *we say* t *is left-dense.*

Notation 5: *For all* $t \in T$, *we put* $inf(\phi) = supT, sup(\phi) = infT$, *and we put*

$$T^{\kappa} = \begin{cases} T \setminus (\rho(supT), supT], & \text{if } supT < \infty \\ T, & \text{if } supT = \infty \end{cases}$$

Definition 6: *The graininess function* $\mu: T \longmapsto [0, \infty)$ *is defined by*

$$\mu(t) := \sigma(t) - t$$

Definition 7: *Let* T *be time scale. We say that the function* $g: T \longmapsto \mathbb{R}$ *is rd-continuous provided that* g *is continuous at each right-dense point* $t \in T$ *and whenever* $t \in T$ *is left-dense* $lim_{s \to t^-} g(s)$ *exists and finite number. The set of* rd*-continuous functions* $g: T \longmapsto \mathbb{R}$ *will be denoted* $C_{rd}(T, \mathbb{R})$.

Remark 8: *We note that if* g *is continuous then it is rd-continuous.*

Definition 9: *Let* T *be a time scale and* $g: T \longmapsto \mathbb{R}$ *is a function with* $t \in T^{\kappa}$, *then we define* $g^{\Delta}(t)$ *to be the number (provided it exists) with the property that given any* $\varepsilon > 0$, *there is neighborhood* U *of* t *such that*

$$|(g(\sigma(t)) - g(s)) - g^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon(\sigma(t) - s)$$

for all $s \in U$.

We call $g^{\Delta}(t)$ the delta derivative of $g(t)$ at t and we say that g is deltadifferentiale at t.

The following result is due to [14] and can be found in [5] and [6].

Theorem 10: *Let T be a time scale and $g: T \longmapsto \mathbb{R}$ is a function with $t \in T^{\kappa}$. Then, we have the following properties:*

(i) If g is continuous at t, then g is delta-differentiale at t.

(ii) If g is continuous at t and t is right-scattered, then g is delta-differentiale at t with

$$g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}.$$

(iii) If g is delta-differentiale at t and right-dense, then

$$g^{\Delta}(t) = \lim_{t \to s} \frac{g(t) - g(s)}{t - s}.$$

(iv) If g is delta-differentiale at t then

$$g^{\sigma}(t) = g(t) + \mu(t) g^{\Delta}(t),$$

Where:

$$g^{\sigma} = g \circ \sigma.$$

The following result can be found in [6].

Theorem 11: *(Mean value theorem) Let g be a contenuous function on $[a, b]_T$ and differentiable on $[a, b)_T$. Then there exists $s, t \in [a, b)_T$ such that*

$$g^{\Delta}(s) \leq \frac{g(b) - g(a)}{b - a} \leq g^{\Delta}(t).$$

Definition 12: *Let T be a time scale and $g: T \longmapsto \mathbb{R}$ is a function. The function*

$G: T^{\kappa} \longmapsto \mathbb{R}$ *is called antiderivative of* g *if* $G^{\Delta}(t) = g(t)$ *for all* $t \in T^{\kappa}$*.*

In this case we define the integral of g *by*

$$\int_{t_1}^{t_2} g(s)\Delta s = G(t_2) - G(t_1), \text{ for } t_1, t_2 \in T.$$

The following theorem is imediate consequence of theorem 2.8 in [10].

Proposition 13: *Let* T *be a time scale and* g, h *are a mapping from* T *to* $\mathbb{R}$ *suth that* $g^{\Delta}(t) = h(t)$*, then we have the following propreties*

(i) If g is a rd-continuous, then g has the antiderivate $h: t \to \int_s^t g(s)\Delta s$, $s, t \in T$.

(ii) If the sequence $(g_n)_{n\in\mathbb{N}}$ of r-continuous functions $T \to \mathbb{R}$ converge uniformly on $[s, t]$ to the rd-continuous function g, then

$$\lim_{n\to+\infty} \int_{\tau}^{s} g_n(t)\Delta t = \int_{\tau}^{s} g(t)\Delta t.$$

Remark 14: *In this paper, we define*

$$D := \left\{ g \in C^1([a, \sigma^2(b)]_T) : \left(\varphi_p(g^{\Delta})\right)^{\Delta} \in C[a, b]_T \right\}.$$

3 PRELIMINARY RESULTS

In this section, we give some preliminary results that will be used in the remainder of this paper.

We consider the following problem

$$\begin{cases} -\left(\varphi_p(u^{\Delta})\right)^{\Delta} = -\hat{h}(t, u^{\sigma}), t \in [a, b]_T, \\ c_1 u(a) - c_2 u^{\Delta}(a) = A, \\ d_1 u(\sigma^2(b)) + d_2 u^{\Delta}(\sigma^2(b)) = B, \end{cases} \quad (2)$$

Where:

$\hat{h}: [a, b]_T \times \mathbb{R} \to \mathbb{R}$ is a continuous function and strictly increasing in its second variable,

c_1, c_2, d_1 and d_2 are a positive real numbers, A and B are real numbers.

Lemma 15: *(Weak comparison principle) and*

Let u_1, u_2 *such that* $u_i \in C^1_{rd}([a,\sigma^2(b)]_T), \varphi_p\left(u_i^\Delta\right) \in C^1_{rd}([a,\sigma(b)]_T), i = 1,2,$

$$\begin{cases} -\left(\varphi_p(u_1^\Delta)\right)^\Delta + \hat{h}(t,u_1^\sigma) \le -\left(\varphi_p(u_2^\Delta)\right)^\Delta + \hat{h}(t,u_2^\sigma), t \in [a,b]_T \\ c_1u_1(a) - c_2u_1^\Delta(a) \le c_1u_2(a) - c_2u_2^\Delta(a) \\ d_1u_1(\sigma^2(b)) + d_2u_1^\Delta(\sigma(b)) \le d_1u_2(\sigma^2(b)) + d_2u_2^\Delta(\sigma(b)) \end{cases} \#(3)$$

then $u_1(t) \le u_2(t)$, *for all* $t \in [a,\sigma^2(b)]_T$.

Proof.Using a proof similar to that lemme 3.1 in [??] . So it is omited.■

Definition 16: We say that α is lower solution of (1) if

(i) $\alpha \in D.$

(ii) $$\begin{cases} -\left(\varphi_p(\alpha^\Delta)\right)^\Delta \le -\hat{h}(t,\alpha^\sigma), t \in [a,b]_T, \\ c_1\alpha(a) - c_2\alpha^\Delta(a) \le A, \\ d_1\alpha(\sigma^2(b)) + d_2\alpha^\Delta(\sigma^2(b)) \le B. \end{cases}$$

Definition 17: *We say that* β *is upper solution of (1) if*

(i) $\beta \in D.$

(ii) $$\begin{cases} -\left(\varphi_p(\beta^\Delta)\right)^\Delta \ge -\hat{h}(t,\beta^\sigma), t \in [a,b]_T, \\ c_1\beta(a) - c_2\beta^\Delta(a) \ge A, \\ d_1\beta(\sigma^2(b)) + d_2\beta^\Delta(\sigma^2(b)) \ge B. \end{cases}$$

Theorem 18: *Assume that* α *and* β *are lower and upper solutions of (1) such that* $\alpha(t) \le \beta(t), \forall t \in [a,\sigma^2(b)]_T$, *then the problem (1) admits unique solution* $u \in D$ *such that*

$$\alpha(t) \le u(t) \le \beta(t), \forall t \in [a,\sigma^2(b)]_T.$$

Proof.Using a proof similar to that of Theorem 3.1 in [1], we can prove that the problem (1) admits at least one solution and by Lemma 1, it follow that this problem admits a unique solution.■

4 MAIN RESULTS

In this section, we state and prove our main result.

On the nonlinearity f and g, we shall impose the following conditions:

$(H)_1 h_1: \mathbb{R} \longmapsto \mathbb{R}$ contunious and increasing function such that $u \longmapsto f(t,u,v) + h_1(u)$ is increasing for all $t \in [a,b]_T$ and $u \in \mathbb{R}$.

$(H)_2 h_2: \mathbb{R} \longmapsto \mathbb{R}$ contunious and increasing function such that $v \longmapsto g(t,u,v) + h_2(v)$ is increasing for all $t \in [a,b]_T$ and $v \in \mathbb{R}$.

Now, we consider the following system

$$\begin{cases} -\left(\varphi_p(u^\Delta)\right)^\Delta = f(t,u^\sigma,v^\sigma), t \in [a,b]_T, \\ -\left(\varphi_q(u^\Delta)\right)^\Delta = g(t,u^\sigma,v^\sigma), t \in [a,b]_T, \\ c_1 u(a) - c_2 u^\Delta(a) = L_1(u), \\ c_3 u(\sigma^2(b)) + c_4 u^\Delta(\sigma^2(b)) = L_2(u), \\ d_1 v(a) - d_2 v^\Delta(a) = L_3(u), \\ d_3 v(\sigma^2(b)) + d_4 v^\Delta(\sigma^2(b)) = L_4(u), \end{cases} \qquad (4)$$

Where:

$\varphi_p(u) = |u|^{p-2} \cdot u$, such that $p > 1$ and $q > 1, c_i, d_i \in \mathbb{R}_+$, for all $i = 0,1,3,4, f: [a,b]_T \times \mathbb{R}^2 \longmapsto \mathbb{R}$ and $g: [a,b]_T \times \mathbb{R}^2 \longmapsto \mathbb{R}$ are continuous functions and $L_i: C([a,\sigma^2(b)]_T) \to \mathbb{R}$ continuous and inreasing functions for all $i = 0,1,3,4$.

4.1 EXISTENCE OF EXTRIMALS SOLUTIONS FOR INCREASING QUASIMONOTOMES SYSTEMS

In this subsection we supose the follwing hypothses:

(H_3): f is increasing of v and g is increasing of u.

Definition 19: *We say that* (u,v) *is a solution of (3) if*

(i) $(u,v) \in D^2$.

(ii) (u,v) *satisfy (3).*

Definition 20: *We say that* $(\underline{u},\underline{v})$ *is a lower solution of* (3) *if*

(i) $(\underline{u},\underline{v}) \in D^2$.

(ii) $\begin{cases} -\left(\varphi_p\left(\underline{u}^{\Delta}\right)\right)^{\Delta} \leq f\left(t, \underline{u}^{\sigma}, \bar{v}^{\sigma}\right), t \in [a,b]_T, \\ -\left(\varphi_q\left(\underline{v}^{\Delta}\right)\right)^{\Delta} \leq g\left(t, \underline{v}^{\sigma}, \underline{v}^{\sigma}\right), t \in [a,b]_T, \\ c_1\underline{u}(a) - c_2\underline{u}^{\Delta}(a) \leq L_1\left(\underline{u}\right), \\ c_3\underline{u}(\sigma^2(b)) + c_4\underline{u}^{\Delta}(\sigma^2(b)) \leq L_2\left(\underline{u}\right), \\ d_1\underline{v}(a) - d_2\underline{v}^{\Delta}(a) \leq L_3\left(\underline{v}\right), \\ d_3\underline{v}(\sigma^2(b)) + d_4\underline{v}^{\Delta}(\sigma^2(b)) \leq L_4\left(\underline{v}\right), \end{cases}$

Definition 21: *We say that $(\bar{u}, \bar{v})$ is an upper solution of (3) if*

(i) $(\bar{u}, \bar{v}) \in D^2$.

(ii) $\begin{cases} -\left(\varphi_p(\bar{u}^{\Delta})\right)^{\Delta} \geq f(t, \bar{u}^{\sigma}, \bar{v}^{\sigma}), t \in [a,b]_T, \\ -\left(\varphi_q(\bar{v}^{\Delta})\right)^{\Delta} \geq g(t, \bar{v}^{\sigma}, \bar{v}^{\sigma}), t \in [a,b]_T, \\ c_1\bar{u}(a) - c_2\bar{u}^{\Delta}(a) \geq L_1(\bar{u}), \\ c_3\bar{u}(\sigma^2(b)) + c_4\bar{u}^{\Delta}(\sigma^2(b)) \geq L_2(\bar{u}), \\ d_1\bar{v}(a) - d_2\bar{v}^{\Delta}(a) \geq L_3(\bar{v}), \\ d_3\bar{v}(\sigma^2(b)) + d_4\bar{v}^{\Delta}(\sigma^2(b)) \geq L_4(\bar{v}), \end{cases}$

Theorem 22: *Let $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are the lower and upper solutions respectively of system (3) such that $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$ in $[a, \sigma^2(b)]_T$, assume that $(H_i)_i = 1,2,3$ holds. Then the system (3) has a maximal solution $(u^\star, v^\star)$ and minimal solution $(u_\star, v_\star)$ such that for every (u, v) of (3) with $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq u \leq \bar{v}$ in $[a, \sigma^2(b)]_T$ we have*

$\underline{u} \leq u_\star \leq u \leq u^\star \leq \bar{u}$ *and* $\underline{v} \leq v_\star \leq v \leq v^\star \leq \bar{v}$ *in* $[a,, \sigma^2(b)]_T$.

Proof.The proof will be given in several steps.

We take $\underline{u}_0 = \underline{u}, \underline{v}_0 = \underline{v}$ and define the sequences $\left(\underline{u}_n\right)_{n\in\mathbb{N}^*}$ and $\left(\underline{v}_n\right)_{n\in\mathbb{N}^*}$ by

$$\begin{cases} -\left(\varphi_p\left(\underline{u}_{n+1}^{\Delta}\right)\right)^{\Delta} + h_1\left(\underline{u}_{n+1}^{\sigma}\right) = f\left(t, \underline{u}_n^{\sigma}, \underline{v}_n^{\sigma}\right) + h_1\left(\underline{u}_n^{\sigma}\right), t \in [a,b]_T, \\ c_1\underline{u}_{n+1}(a) - c_2\underline{u}_{n+1}^{\Delta}(a) = L_1\left(\underline{u}_n\right), \\ c_3\underline{u}_{n+1}(\sigma^2(b)) + c_4\underline{u}_{n+1}^{\Delta}\left(\sigma(b)\right) = L_2\left(\underline{u}_n\right), \end{cases} \quad (5)$$

And

$$\begin{cases} -\left(\varphi_q\left(\underline{v}_{n+1}^{\Delta}\right)\right)^{\Delta} + h_2\left(\underline{v}_{n+1}^{\sigma}\right) = g\left(t, \underline{u}_n^{\sigma}, \underline{v}_n^{\sigma}\right) + h_2\left(\underline{v}_n^{\sigma}\right), t \in [a,b]_T, \\ d_1 \underline{v}_{n+1}(a) - d_2 \underline{v}_{n+1}^{\Delta}(a) = L_3\left(\underline{v}_n\right), \\ d_3 \underline{v}_{n+1}(\sigma^2(b)) + d_4 \underline{v}_{n+1}^{\Delta}\left(\sigma(b)\right) = L_4\left(\underline{v}_n\right), \end{cases} \quad (6)$$

Likewise we take $\bar{u}_0 = \bar{u}, \bar{v}_0 = \bar{v}$ and define the sequences $(\bar{u}_n)_{n\in\mathbb{N}^\star}$ and $(\bar{v}_n)_{n\in\mathbb{N}^\star}$ by

$$\begin{cases} -\left(\varphi_p\left(\bar{u}_{n+1}^{\Delta}\right)\right)^{\Delta} + h_1(\bar{u}_{n+1}^{\sigma}) = f(t, \bar{u}_n^{\sigma}, \bar{v}_n^{\sigma}) + h_1(\bar{u}_n^{\sigma}), t \in [a,b]_T, \\ c_1 \bar{u}_{n+1}(a) - c_2 \bar{u}_{n+1}^{\Delta}(a) = L_1(\bar{u}_n), \\ c_3 \bar{u}_{n+1}(\sigma^2(b)) + c_4 \bar{u}_{n+1}^{\Delta}\left(\sigma(b)\right) = L_2(\bar{u}_n), \end{cases} \quad (7)$$

And

$$\begin{cases} -\left(\varphi_q\left(\bar{v}_{n+1}^{\Delta}\right)\right)^{\Delta} + h_2(\bar{v}_{n+1}^{\sigma}) = g(t, \bar{u}_n^{\sigma}, \bar{v}_n^{\sigma}) + h_2(\bar{v}_n^{\sigma}), t \in [a,b]_T, \\ d_1 \bar{v}_{n+1}(a) - d_2 \bar{v}_{n+1}^{\Delta}(a) = L_3(\bar{v}_n), \\ d_3 \bar{v}_{n+1}(\sigma^2(b)) + d_4 \bar{v}_{n+1}^{\Delta}\left(\sigma(b)\right) = L_4(\bar{v}_n), \end{cases} \quad (8)$$

Step 1: We prove that

$\forall n \in \mathbb{N}: \underline{u} \leq \underline{u}_n \leq \bar{u}$ and $\underline{v} \leq \underline{v}_n \leq \bar{v}$, in $[a, (\sigma^2(b)]_T$.

i) For $n = 0$, we have:

$\underline{u} \leq \underline{u}_0$ alors $\underline{u} \leq \underline{u}_0 \leq \bar{u}$ and $\underline{v} \leq \underline{v}_0 \leq \bar{v}$.

ii) Assume for fixed $n > 1$, we have :

$\underline{u} \leq \underline{u}_n \leq \bar{u}$ and $\underline{v} \leq \underline{v}_n \leq \bar{v}$.

and we show that:

$\underline{u} \leq \underline{u}_{n+1} \leq \bar{u}$ and $\underline{v} \leq \underline{v}_{n+1} \leq \bar{v}$.

We have

$$\left(\varphi_p\left(\underline{u}_{n+1}^{\Delta}\right)\right)^{\Delta}+h_1\left(\underline{u}_{n+1}^{\sigma}\right)+\left(\varphi_p\left(\underline{u}^{\Delta}\right)\right)^{\Delta}-h_1\left(\underline{u}^{\sigma}\right),\geq f\left(t,\underline{u}_n^{\sigma},\underline{v}_n^{\sigma}\right)+h_1\left(\underline{u}_n^{\sigma}\right)-f\left(t,\underline{u}^{\sigma},\underline{v}^{\sigma}\right)-h_1\left(\underline{u}_n^{\sigma}\right),$$
$$\geq f\left(t,\underline{u}_n^{\sigma},\underline{v}^{\sigma}\right)+h_1\left(\underline{u}_n^{\sigma}\right)-f\left(t,\underline{u}^{\sigma},\underline{v}^{\sigma}\right)-h_1\left(\underline{u}_n^{\sigma}\right),$$
$$\geq 0.$$

Then

$$-\left(\varphi_p\left(\underline{u}_{n+1}^{\Delta}\right)\right)^{\Delta}+h_1\left(\underline{u}_{n+1}^{\sigma}\right)\geq-\left(\varphi_p\left(\underline{u}^{\Delta}\right)\right)^{\Delta}+h_1\left(\underline{u}^{\sigma}\right). \qquad (9)$$

Other hand, we have

$$c_1\underline{u}_{n+1}(a)-c_2\underline{u}_{n+1}^{\Delta}(a)-c_1\underline{u}(a)+c_2\underline{u}^{\Delta}(a)=L_1\left(\underline{u}_n\right)-L_1\left(\underline{u}\right),$$
$$\geq 0.$$

Then

$$c_1\underline{u}_{n+1}(a)-c_2\underline{u}_{n+1}^{\Delta}(a)\geq c_1\underline{u}(a)-c_2\underline{u}^{\Delta}(a). \qquad (10)$$

De meme, we have

$$c_3\underline{u}_{n+1}(\sigma^2(b))+c_4\underline{u}_{n+1}^{\Delta}\left(\sigma(b)\right)\geq c_3\underline{u}\left(\sigma^2(b)\right)+c_4\underline{u}^{\Delta}\left(\sigma(b)\right). \qquad (11)$$

By (8),(9) and (10) and we use the Lemme 1, we obtain

$\underline{u}\leq\underline{u}_{n+1}$ in $[a,(\sigma^2(b)]_T$.

With similar method, we can proof

$\underline{u}_{n+1}\leq\bar{u}$ in $[a,(\sigma^2(b)]_T$.

And

$\underline{v}\leq\underline{v}_{n+1}\leq\bar{v}$ in $[a,(\sigma^2(b)]_T$.

In conclusion

$\underline{u} \leq \underline{u}_{n+1} \leq \bar{u}$ in $\underline{v} \leq \underline{v}_{n+1} \leq \bar{v}$ in $[a, (\sigma^2(b)]_T$.

Step 2: By the similar method of step 1, we proof

$\forall n \in \mathbb{N}: \underline{u}_n \leq \underline{u}_{n+1}$ and $\underline{v}_n \leq \underline{v}_{n+1}$, in $[a, (\sigma^2(b)]_T$.

Step 3: By the similar method of step 1, we proof

$\underline{u} \leq \bar{u}_{n+1} \leq \bar{u}_n \leq \bar{u}$ and $\underline{v} \leq \bar{v}_{n+1} \leq \bar{v}_n \leq \bar{v}$ in $[a, (\sigma^2(b)]_T$.

Step 4: There exists two positives constants C_1 and C_2 independents of $n \in \mathbb{N}$, such that

$\|\bar{u}_n^{\Delta}\|_0 \leq C_1$ and $\|\bar{v}_n^{\Delta}\|_0 \leq C_2$, for all $n \in \mathbb{N}$.

1) Let $n \in \mathbb{N}$, and $t \in [a, (\sigma(b)]_T$, since $\bar{u}_n$ is continuous in $[a, (\sigma^2(b)]_T$ and $\bar{u}_n^{\Delta}$ is continuous in $[a, (\sigma(b))_T$, then by Theorem 11, there exists $t_n, s_n \in [a, (\sigma^2(b))_T$, such that

$$\bar{u}_n^{\Delta}(s_n) \leq \frac{\bar{u}_n\left(\sigma^2(b)\right) - \bar{u}_n(a)}{\sigma^2(b) - a} \leq \bar{u}_n^{\Delta}(t_n).$$

Then we have

$$-\varphi_p\left(\bar{u}_{n+1}^{\Delta}(t)\right) = -\varphi_p\left(\bar{u}_{n+1}^{\Delta}(s_n)\right) + \int_{s_n}^{r} \left(f\left(s, \bar{u}_n^{\sigma}(s), \bar{v}_n^{\sigma}(s)\right) + h_1\left(\bar{u}_n^{\sigma}(s)\right) - h_1\left(\bar{u}_{n+1}^{\sigma}(s)\right)\right) \Delta s,$$
$$\geq \frac{1}{(\sigma^2(b) - a)^{p-1}} \varphi_p\left(\bar{u}\left(\sigma^2(b) - \underline{u}(a)\right) + K\right.,$$

Where

$K = (M_1(f) + 2M_2(h))(\sigma^2(b) - a),$

With

$$M_1(f) = \max\{|f(t,u,v)|: t \in [a,b]_T, \underline{u} \le u \le \bar{u} \text{ and } \underline{v} \le v \le \bar{v}\},$$

And

$$M_2(h) = \max\{|h(u)|: u \le u \le \bar{u}\}.$$

Then if we put

$$C' := \frac{1}{(\sigma^2(b)-a)^{p-1}} \varphi_p\big(\bar{u}(\sigma^2(b) - \underline{u}(a)\big) + (M_1(f) + 2M_2(h))(\sigma^2(b) - a),$$

we obtain

$$\forall n \in \mathbb{N}, \forall t \in \Big[a, (\sigma(b)]_T : -\varphi_p\left(\bar{u}_{n+1}^{\Delta}(t)\right) \le C'.$$

Which leads to

$$\forall n \in \mathbb{N}, \forall t \in \Big[a, (\sigma(b)]_T : \bar{u}_{n+1}^{\Delta}(t) \le C'^{\frac{1}{p-1}}. \quad (12)$$

With a similair way

$$\forall n \in \mathbb{N}, \forall t \in \Big[a, (\sigma(b)]_T : -\varphi_p\left(\bar{u}_{n+1}^{\Delta}(t)\right) \ge C'', \quad (13)$$

Where

$$C'' := \frac{1}{(\sigma^2(b)-a)^{p-1}} \varphi_p\big(\underline{u}(\sigma^2(b) - \bar{u}(a)) + (m_1(f) + 2m_2(h))\big(\sigma^(b\big) - a\big),$$

With

$m_1(f) = \min\{|f(t,u,v)|: t \in [a,b]_T, \underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}\}$,

And

$m_2(h) = \min\{|h(u)|: u \leq u \leq \bar{u}\}$.

Then by (12), we obtain

$$\forall n \in \mathbb{N}, \forall t \in \left[a, (\sigma(b)\right]_T : \bar{u}_{n+1}^{\Delta}(t) \geq |C''|^{\frac{2-p}{p-1}} C''. \qquad (14)$$

Now, if we put

$C_1 = \min\left\{C^{\frac{1}{p-1}}, |C''|^{\frac{2-p}{p-1}} C'', \|\bar{u}_n^{\Delta}\|_0\right\}$.

then by (11) and (13), we obtain

$\forall n \in \mathbb{N}: \|\bar{u}_n^{\Delta}\|_0 \leq C'$.

In a similar way, we proof that

$\forall n \in \mathbb{N}: \|\bar{v}_n^{\Delta}\|_0 \leq C_1'$.

Step 5: By the similar proof of Step 4, we schow that there exists two positives constants C_2 and C_3 independents of $n \in \mathbb{N}$, such that

$\|\underline{u}_n^{\Delta}\|_0 \leq C_2$ and $\|\underline{v}_n^{\Delta}\|_0 \leq C_3$, for all $n \in \mathbb{N}$.

Step 6: The sequences of functions $(\bar{u}_n^{\Delta})_{n\in\mathbb{N}}$ and $(\bar{v}_n^{\Delta})_{n\in\mathbb{N}}$ are equicontinuouos on $[a, \sigma(b)]_T$.

Let $\varepsilon > 0$ and $t, s \in [a, \sigma(b)]_T$ such that $t < s$, then for each $n \in \mathbb{N}$, we have

$$\left|\varphi_p\left(\bar{u}_{n+1}^{\Delta}(s)\right)-\varphi_p\left(\bar{u}_{n+1}^{\Delta}(t)\right)\right| \leq K_1|s-t|.$$

Then, if we choose

$$|s-r| \leq \frac{\varepsilon}{K_1+1},$$

we obtain

$$\varphi_p\left(\bar{u}_{n+1}^{\Delta}(s)\right)-\left|\varphi_p\left(\bar{u}_{n+1}^{\Delta}(r)\right)\right| \leq \varepsilon.$$

Consequentely, the sequence $\left(\varphi_p\left(\bar{u}_{n+1}^{\Delta}(s)\right)_{n\in\mathbb{N}}\right.$ is equicontinuouos on $[a,\sigma(b)]_T$. Sach us φ_p^{-1} is the homeomorphisme of $\mathbb{R}$ onto $\mathbb{R}$, we deduce that

$$\left|\bar{u}_n^{\Delta}(s)-\bar{u}_n^{\Delta}(t)\right| = \left|\varphi_p^{-1}\left(\varphi_p(\bar{u}_n^{\Delta}(s))\right)-\varphi_p^{-1}\left(\varphi_p(\bar{u}_n^{\Delta}(r))\right)\right|.$$

Then the sequence $(\bar{u}_n^{\Delta})_{n\in\mathbb{N}}$ is equicontinuouos on $[a,\sigma(b)]_T$.

By the similar way, we chow that the sequence $(\bar{v}_n^{\Delta})_{n\in\mathbb{N}}$ is equicontinuouos on $[a,\sigma(b)]_T$.

Step 7: The sequences of functions $(\underline{u}_n^{\Delta})_{n\in\mathbb{N}}$ and $(\underline{v}_n^{\Delta})_{n\in\mathbb{N}}$ are equicontinuouos on $[a,\sigma(b)]_T$.

The proof is similar of Step 6 .

Step 8: The sequence $(\bar{u}_n,\bar{v}_n)_{n\in\mathbb{N}}$ converge to the maximal solution of (3). By the Step 1, 2 and 6 , we have $(\bar{u}_n)_{n\in\mathbb{N}}$ and $(\bar{v}_n)_{n\in\mathbb{N}}$ are uniformly bounded on $C^1([a,\sigma^2(b)]_T)$ and $(\bar{u}_n^{\Delta},\bar{v}_n^{\Delta})_{n\in\mathbb{N}}$ are equicontinuouos on $[a,\sigma(b)]_T$.

Hence by the Ascoli-Arzila theorem, there exists a subsequences $\left(\bar{u}_{n_j}^{\Delta}\right)_{n_j\in\mathbb{N}}$ of$(\bar{u}_n^{\Delta})_{n\in\mathbb{N}}$ and $\left(\bar{v}_{n_j}^{\Delta}\right)_{n_j\in\mathbb{N}}$ of $(\bar{v}_n^{\Delta})_{n\in\mathbb{N}}$ which converges in $C^1([a,b]_T)$.

Let

$$u:=\lim_{n_j\to+\infty}\bar{u}_{n_j} \text{ and } v:=\lim_{n_j\to+\infty}\bar{v}_{n_j},$$

Then

$$u^{\Delta} = \lim_{n_j \to +\infty} \bar{u}^{\Delta}_{n_j} \text{ and } v^{\Delta} = \lim_{n_j \to +\infty} \bar{v}^{\Delta}_{n_j}.$$

But, by Step 1 the sequences $(\underline{u}_n)_{n\in\mathbb{N}}$ and $(\underline{v}_n)_{n\in\mathbb{N}}$ are decreasing and bounded, then the limits of these sequences exists are denoted u^* and v^* respectly.

Hence we have $u = u^\star$ and $v = v^\star$ and moreover, the wholes sequences converges to $u^\star$ and $v^\star$ rspectly in $C^1_{rd}([a,b]_T)$.

Let $t \in (a,b)_T$, we have

$$-\varphi_p\left(\bar{u}^{\Delta}_{n+1}(t)\right) = -\varphi_p\left(\bar{u}^{\Delta}_{n}(a)\right) + \int_a^t \left(f\left(s, \bar{u}^{\sigma}_n(s), (\bar{v}^{\sigma}_n(s)\right) - h_1\left(\bar{u}^{\sigma}_n(s)\right) - h - 1\left(\bar{u}^{\sigma}_{n+1}(s)\right)\right)\Delta s,$$

And

$$-\varphi_p\left(\bar{v}^{\Delta}_{n+1}(t)\right) = -\varphi_p\left(\bar{v}^{\Delta}_{n}(a)\right) + \int_a^t \left(g\left(s, \bar{u}^{\sigma}_n(s), (\bar{v}^{\sigma}_n(s)\right) - h_2\left(\bar{v}^{\sigma}_n(s)\right) - h - 2\left(\bar{v}^{\sigma}_{n+1}(s)\right)\right)\Delta s,$$

Now, as n tends to $+\infty$, we obtain

$$(f(s, \bar{u}^{\sigma}_n(s), (\bar{v}^{\sigma}_n(s)) - h_1(\bar{u}^{\sigma}_n(s)) - h - 1(\bar{u}^{\sigma}_{n+1}(s))) \to (f(s, \bar{u}^\star(s), (\bar{v}^\star(s)).$$

And

$$(g(s, \bar{u}^{\sigma}_n(s), (\bar{v}^{\sigma}_n(s)) - h_2(\bar{v}^{\sigma}_n(s)) - h - 2(\bar{2}^{\sigma}_{n+1}(s))) \to (g(s, \bar{u}^\star(s), (\bar{v}^\star(s)).$$

Also, we have

$$\exists K_2 > 0; \forall n \in \mathbb{N}; \forall s \in [a,b]_T: \left| f\left(s, \bar{u}_n(s), (\bar{v}_n(s)) + h_1(\bar{u}_n(s)) - h_1\left(\bar{u}_{n+1}^{\Delta}(s)\right)\right)\right| \leq K_2.$$

and

$$\exists K_3 > 0; \forall n \in \mathbb{N}; \forall s \in [a,b]_T: \left| g\left(s, \bar{u}_n(s), (\bar{v}_n(s)) + h_2(\bar{v}_n(s)) - h_2\left(\bar{v}_{n+1}^{\Delta}(s)\right)\right)\right| \leq K_3.$$

Hence, the dominated onvergence theorem of Lebegue implies

$$-\varphi_p\left(u^{\star\Delta}(t)\right) = -\varphi_p\left(u^{\star\Delta}(a)\right) + \int_a^t \left(f\left(s, u^{\star}(s), v^{\star\Delta}(s)\right) + h_1\left(u^{\star}(s)\right) - h_1\left(u^{\star\Delta}(s)\right)\right)\Delta s,$$

And

$$-\varphi_p\left(u^{\star\Delta}(t)\right) = -\varphi_p\left(u^{\star\Delta}(a)\right) + \int_a^t \left(g\left(s, u^{\star}(s), v^{\star\Delta}(s)\right) + h_2\left(v^{\star}(s)\right) - h_2\left(v^{\star\Delta}(s)\right)\right)\Delta s,$$

Thus, we obtain $\forall t \in (a,b)_T$

$$-\varphi_p\left(u^{\star\Delta}(t)\right) = f\left(s, u^{\star}(s), \left(v^{\star\Delta}(\tau)\right)\right), \qquad (15)$$

Thus

$$c_1 u^{\star}(a) - c_2 u^{\star\Delta}(a) = L_1(u^{\star}), \qquad (16)$$

And

$$d_1 u^\star(\sigma^2(b)) + d_2 u^{\star\Delta}(\sigma^2(b)) = L_2(u^\star). \qquad (17)$$

De meme, we have

$$-\varphi_q\left(v^{\star\Delta}(t)\right) = g\left(s, u^\star(s), \left(v^{\star\Delta}(s)\right)\right), \qquad (18)$$

Thus

$$d_1 v^\star(a) - d_2 v^{\star\Delta}(a) = L_3(u^\star), \qquad (19)$$

And

$$d_3 v^\star(\sigma^2(b)) + d_4 v^{\star\Delta}(\sigma^2(b)) = L_4(u^\star). \qquad (20)$$

By (14), (15), (16), (17), (18), and (19) its results that $(u^\star, v^\star)$ is a solution of (3).

Now, we prove that if (u, v) is the another solution of (3) such that $u \le u \le \bar{u}$ and $\underline{v} \le v \le \bar{v}$ in $[a, b]_T$, then $u \le u^\star$ and $v \le v^\star$ in $[a, b]_T$.

Since (u, v) is a lower solution of (3), then by Step 1, we obtain

$$\forall n \in \mathbb{N}: u \le \bar{u}_n \text{ and } v \le \bar{v}_n,$$

letting $n \to +\infty$, we obtain

$$u \le \lim_{n\to+\infty} \bar{u}_n = u^\star \text{ and } v \le \lim_{n\to+\infty} \bar{v}_n = v^\star,$$

wihich mean that $(u^\star, v^\star)$ is a solution of (3).

Step 9: Analagously we shows that $\left(\underline{u}_n, \underline{v}_n\right)_{n\in\mathbb{N}}$ coverges to a minimal solution $(u_\star, v_\star)$ of (3). The proof of our result is complete.

4.2 EXISTENCE OF MINIMAL-MAXIMAL AND MAXIMAL-MINIMAL SOLUTIONS FOR DECREASING QUASIMONOTOMS SYSTEMS

In this subsection we supose the follwing hypothse:

(H_4): f is decreasing according of v and g is decreasing according of u.

Definition 23: *We say that $(u,\bar{u})$ and $(\underline{v},\bar{v})$ are a lower and upper solution of (3) if*

(i) $(\underline{u},\bar{u})$ *and* $(\underline{v},\bar{v}) \in D^2$.

$$(ii)\begin{cases} -\left(\varphi_p\left(\underline{u}^{\Delta}\right)\right)^{\Delta} \le f\left(t,\underline{u}^{\sigma},\bar{v}^{\sigma}\right), t\in[a,b]_T, \\ -\left(\varphi_p\left(\underline{u}^{\Delta}\right)\right)^{\Delta} \ge f\left(t,\bar{u}^{\sigma},\underline{v}^{\sigma}\right), t\in[a,b]_T, \\ -\left(\varphi_q\left(\underline{v}^{\Delta}\right)\right)^{\Delta} \le g\left(t,\bar{u}^{\sigma},\underline{v}^{\sigma}\right), t\in[a,b]_T, \\ -\left(\varphi_q\left(\underline{v}^{\Delta}\right)\right)^{\Delta} \ge g\left(t,\underline{u}^{\sigma},\bar{v}^{\sigma}\right), t\in[a,b]_T, \\ c_1\underline{u}(a)-c_2\underline{u}^{\Delta}(a)\le L_1(\underline{u}), c_3\underline{u}(\sigma^2(b))+c_4\underline{u}^{\Delta}(\sigma^2(b))\le L_2(\underline{u}), \\ d_1\underline{v}(a)-d_2\underline{v}^{\Delta}(a)\le L_3(\underline{v}), d_3\underline{v}(\sigma^2(b))+d_4\underline{v}^{\Delta}(\sigma^2(b))\le L_4(\underline{v}), \\ c_1\bar{u}(a)-c_2\bar{u}^{\Delta}(a)\ge L_1(\bar{u}), c_3\bar{u}(\sigma^2(b))+c_4\bar{u}^{\Delta}(\sigma^2(b))\ge L_2(\bar{u}), \\ d_1\bar{v}(a)-d_2\bar{v}^{\Delta}(a)\ge L_3(\bar{v}), d_3\bar{v}(\sigma^2(b))+d_4\bar{v}^{\Delta}(\sigma^2(b))\ge L_4(\bar{v}). \end{cases}$$

Theorem 24: *Assume that these hypothses $(H_1),(H_2)$ and (H_4) are holds and let $(\underline{u},\bar{u})$ and $(\underline{v},\bar{v})$ are the lower and upper solutions respectively of system (3) such that $\underline{u}\le\bar{u}$ and $\underline{v}\le\bar{v}$ in $[a,\sigma^2(b)]_T$. Then the system (3) has a maximalminimal solution $(u^{\star},v_{\star})$ and minimal-mximal solution $(u_{\star},v^{\star})$ such that for every (u,v) of (3) with $(\underline{u},\underline{v})\le(u,v)\le(\bar{u},\bar{v})$, we have*

$$(\underline{u},\underline{v})\le(u_{\star},v_{\star})\le(u,v)\le(\bar{u},\bar{v})\le(u^{\star},v^{\star}).$$

Proof. We take $\bar{u}_0=\bar{u},\underline{v}_0=\underline{v}$ and define the sequences $(\bar{u}_n)_{n\in\mathbb{N}^{\star}}$ and $(\underline{v}_n)_{n\in\mathbb{N}^{\star}}$ by

$$\begin{cases} -\left(\varphi_p\left(\bar{u}_{n+1}^{\Delta}\right)\right)^{\Delta}+h_1(\bar{u}_{n+1}^{\sigma})=f\left(t,\bar{u}_n^{\sigma},\underline{v}_n^{\sigma}\right)+h_1(\bar{u}_n^{\sigma}), t\in[a,b]_T, \\ c_1\bar{u}_{n+1}(a)-c_2\bar{u}_{n+1}^{\Delta}(a)=L_1\left(\underline{u}_n\right), \\ c_3\bar{u}_{n+1}(\sigma^2(b))+c_4\bar{u}_{n+1}^{\Delta}\left(\sigma(b)\right)=L_2(\bar{u}_n), \end{cases} \qquad (21)$$

And

$$\begin{cases}-\left(\varphi_q\left(\underline{v}_{n+1}^{\Delta}\right)\right)^{\Delta}+h_2\left(\underline{v}_{n+1}^{\sigma}\right)=g\left(t,\bar{u}_n^{\sigma},\underline{v}_n^{\sigma}\right)+h_2\left(\underline{v}_n^{\sigma}\right), t\in[a,b]_T,\\ d_1\underline{v}_{n+1}(a)-d_2\underline{v}_{n+1}^{\Delta}(a)=L_3\left(\underline{v}_n\right),\\ d_3\underline{v}_{n+1}(\sigma^2(b))+d_4\underline{v}_{n+1}^{\Delta}\left(\sigma(b)\right)=L_4\left(\underline{v}_n\right),\end{cases} \tag{22}$$

Likewise we take $\underline{u}_0=\underline{u},\bar{v}_0=\bar{v}$ and define the sequences $\left(\underline{u}_n\right)_{n\in\mathbb{N}^*}$ and $(\bar{v}_n)_{n\in\mathbb{N}^*}$ by

$$\begin{cases}-\left(\varphi_p\left(\underline{u}_{n+1}^{\Delta}\right)\right)^{\Delta}+h_1\left(\underline{u}_{n+1}^{\sigma}\right)=f\left(t,\underline{u}_n^{\sigma},\bar{v}_n^{\sigma}\right)+h_1\left(\underline{u}_n^{\sigma}\right), t\in[a,b]_T,\\ c_1\underline{u}_{n+1}(a)-c_2\underline{u}_{n+1}^{\Delta}(a)=L_1(\bar{u}_n),\\ c_3\underline{u}_{n+1}(\sigma^2(b))+c_4\bar{u}_{n+1}^{\Delta}\left(\sigma(b)\right)=L_2\left(\underline{u}_n\right),\end{cases} \tag{23}$$

And

$$\begin{cases}-\left(\varphi_q\left(\bar{v}_{n+1}^{\Delta}\right)\right)^{\Delta}+h_2(\bar{v}_{n+1}^{\sigma})=g\left(t,\underline{u}_n^{\sigma},\bar{v}_n^{\sigma}\right)+h_2(\bar{v}_n^{\sigma}), t\in[a,b]_T,\\ d_1\bar{v}_{n+1}(a)-d_2\bar{v}_{n+1}^{\Delta}(a)=L_3(\bar{v}_n),\\ d_3\bar{v}_{n+1}(\sigma^2(b))+d_4\bar{v}_{n+1}^{\Delta}\left(\sigma(b)\right)=L_4(\bar{v}_n),\end{cases} \tag{24}$$

The rest of the proof is similar us Theorem 21.■

4.3 EXISTENCE OF SOLUTIONS FOR MIXED QUASIMONOTOMS SYSTEMS

In this subsection we supose the follwing hypothesis:

(H_5): f is increasing in v and g is decreasing in u.

Definition 25: *We say that* $(\underline{u},\bar{u})$ *and* $(\underline{v},\bar{v})$ *are a lower and upper pair solution of (3) if*

(i) $(\underline{u},\bar{u})$ *and* $(\underline{v},\bar{v})\in D^2$.

$$(ii)\begin{cases}-\left(\varphi_p(\underline{u}^{\Delta})\right)^{\Delta} \leq f(t,\underline{u}^{\sigma},\underline{v}^{\sigma}), t\in[a,b]_T,\\ -\left(\varphi_p(\bar{u}^{\Delta})\right)^{\Delta} \geq f(t,\bar{u}^{\sigma},\bar{v}^{\sigma}), t\in[a,b]_T,\\ -\left(\varphi_q(\underline{v}^{\Delta})\right)^{\Delta} \leq g(t,\bar{u}^{\sigma},\underline{v}^{\sigma}), t\in[a,b]_T,\\ -\left(\varphi_q(\bar{v}^{\Delta})\right)^{\Delta} \geq g(t,\underline{u}^{\sigma},\bar{v}^{\sigma}), t\in[a,b]_T,\\ c_1\underline{u}(a)-c_2\underline{u}^{\Delta}(a)\leq L_1(\underline{u}), c_3\underline{u}(\sigma^2(b))+c_4\underline{u}^{\Delta}(\sigma^2(b))\leq L_2(\underline{u}),\\ d_1\underline{v}(a)-d_2\underline{v}^{\Delta}(a)\leq L_3(\underline{v}), d_3\underline{v}(\sigma^2(b))+d_4\underline{v}^{\Delta}(\sigma^2(b))\leq L_4(\underline{v}),\\ c_1\bar{u}(a)-c_2\bar{u}^{\Delta}(a)\geq L_1(\bar{u}), c_3\bar{u}(\sigma^2(b))+c_4\bar{u}^{\Delta}(\sigma^2(b))\geq L_2(\bar{u}),\\ d_1\bar{v}(a)-d_2\bar{v}^{\Delta}(a)\geq L_3(\bar{v}), d_3\bar{v}(\sigma^2(b))+d_4\bar{v}^{\Delta}(\sigma^2(b))\geq L_4(\bar{v}),\end{cases}$$

We take $u_0=\bar{u}, u_1=\underline{u}$ and $v_0=\bar{v}, v_1=\underline{v}$ and define the sequences $(\bar{u}_n)_{n\in\mathbb{N}^*}$ and $(\underline{v}_n)_{n\in\mathbb{N}^*}$ by

$$\begin{cases}-\left(\varphi_p(u_{n+2}^{\Delta})\right)^{\Delta}+h_1(u_{n+2}^{\sigma})=f(t,u_n^{\sigma},v_n^{\sigma})+h_1(u_n^{\sigma}), t\in[a,b]_T,\\ c_1u_{n+2}(a)-c_2u_{n+2}^{\Delta}(a)=L_1(u_n),\\ c_3u_{n+2}(\sigma^2(b))+c_4u_{n+2}^{\Delta}(\sigma(b))=L_2(u_n),\end{cases} \quad (25)$$

And

$$\begin{cases}-\left(\varphi_q(v_{n+2}^{\Delta})\right)^{\Delta}+h_2(v_{n+2}^{\sigma})=g(t,u_{n+1}^{\sigma},v_n^{\sigma})+h_2(v_n^{\sigma}), t\in[a,b]_T,\\ d_1v_{n+2}(a)-d_2v_{n+2}^{\Delta}(a)=L_3(v_n),\\ d_3v_{n+2}(\sigma^2(b))+d_4v_{n+2}^{\Delta}(\sigma(b))=L_4(v_n),\end{cases} \quad (26)$$

and we have the following result.

Theorem 26: *Assume that these hypothses* $(H_1),(H_2),(H_3) and (H_5)$ *are holds and let* $(\underline{u},\bar{u})$ *and* $(\underline{v},\bar{v})$ *are the lower and upper pair solutions respectively of system (3) within the meaning of definition 24 such that* $\underline{u}\leq\bar{u}$ *and* $\underline{v}\leq\bar{v}$ *in* $[a,\sigma^2(b)]_T$. *Then the the sequences* $(u_n)_{n\in\mathbb{N}^*}$ *and* $(v_n)_{n\in\mathbb{N}^*}$ *are monotonous and verify*

$$\underline{u}=u_1\leq u_3\leq\cdots\leq(u_{2n+1}\leq u_{2n}\leq\cdots\leq u_2\leq u_0=\bar{u},$$
$$\underline{v}=v_1\leq v_3\leq\cdots\leq v_{2n+1}\leq v_{2n}\leq\cdots\leq v_2\leq v_0=\bar{u}.$$

Proof.The proof is similar us proof of Step 1 of theorem 21. ■According to the previous theorem the sequences $((u_{2n}, v_{2n})_{n\in\mathbb{N}}$ and $((u_{2n+1}, v_{2n+1})_{n\in\mathbb{N}}$ are monotonous and bounded. Consequently we put by definition

$$\lim_{n\to+\infty}(u_{2n}, u_{2n}) = (u_{**}, u_{**}),$$

And

$$\lim_{n\to+\infty}(u_{2n+1}, u_{2n+1}) = (u^{**}, u^{**}),$$

and we have

$$\underline{u} \leq (u_{**} \leq u^{**} \leq \bar{u},$$

And

$$\underline{v} \leq (v_{**} \leq v^{**} \leq \bar{u}.$$

Using similar reasoning us theorem 21 , we shows the following result.

Theorem 27: *Assume that these hypothses $(H_1), (H_2), (H_3)$ and (H_5) are holds and let $(\underline{u}, \bar{u})$ and $(\underline{v}, \bar{v})$ are the lower and upper pair solutions respectively of system (3) within the meaning of definition 24 such that $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$ in $[a, \sigma^2(b)]_T$, then the the sequences $((u_{2n}, v_{2n})_{n\in\mathbb{N}}$ and $((u_{2n+1}, v_{2n+1})_{n\in\mathbb{N}}$ converges to (u_{**}, u_{**}) and (u^{**}, u^{**}) respectly which are quasi-solution for the problem 3.*

Proof.The proof is similar us proof of Step 1 of theorem 21.■

5 APPLICATIONS

In this section, we apply the previous result to the following problem.

5.1 EXEMPLE 1

$$\begin{cases} -\left(\varphi_p(u^\Delta)\right)^\Delta = u^{\lambda_1}(\sigma(t)) - u^{k_1}(\sigma(t)) + u^{\lambda_2}(\sigma(t))v^{k_2}(\sigma(t)) + h_1(t) \text{ in } [0,10]_T, \\ -\left(\varphi_q(u^\Delta)\right)^\Delta = u^{\lambda_3}(\sigma(t)) - u^{k_3}(\sigma(t)) + u^{\lambda_4}(\sigma(t))v^{k_4}(\sigma(t)) + h_2(t) \text{ in } [0,10]_T, \\ u(0) - u^\Delta(0) = \int_0^{\sigma^2(10)} g_1(s)u(s)\Delta s, u\left(\sigma^2(10)\right) + u^\Delta\left(\sigma(10)\right) = \int_0^{\sigma^2(10)} g_2(s)u(s)\Delta s, \\ v(0) - v^\Delta(0) = \int_0^{\sigma^2(10)} g_3(s)u(s)\Delta s, v\left(\sigma^2(10)\right) + v^\Delta\left(\sigma(10)\right) = \int_0^{\sigma^2(10)} g_2(s)v(s)\Delta s, \end{cases} \quad (27)$$

Where:

$\lambda_i > 0$ and $k_i > 0$ for all $i = 1,2,3,4$ wich $k_1 > max(\lambda_1, \lambda_2 + k_2)$, and $k_3 > max(\lambda_3, \lambda_4 + k_4)$ and $h: [0,10]_T \to \mathbb{R}_+$are a contunious functions for $i = 1,2$ and $\int_0^{\sigma^2(10)} g_i(s)u(s)\Delta s < 1$ for all $i = 1,2,3,4$.

First of all it is easy to verify that the is increasing quasi-monotone system and the hypothesies $(H_1), (H_2)$ and (H_3) are holds.

We will study two cases.

Case 1: If $T = \mathbb{R}$, we have

$$\begin{cases} -\left(\varphi_p(u')\right)' = u^{\lambda_1}(t) - u^{k_1}(t) + u^{\lambda_2}(t)v^{k_2}(t) + h_1(t) \text{ in } [0,10], \\ -\left(\varphi_q(u')\right)' = u^{\lambda_3}(t) - u^{k_3}(t) + u^{\lambda_4}(t)v^{k_4}(t) + h_2(t) \text{ in } [0,10], \\ u(0) - u'(0) = \int_0^{10} g_1(s)u(s)ds, u(10) + u'(10) = \int_0^{10} g_2(s)u(s)ds, \\ v(0) - v'(0) = \int_0^{10} g_3(s)u(s)ds, v(10) + v'(10) = \int_0^{10} g_2(s)v(s)ds, \end{cases} \quad (28)$$

Then we have the result.

Theorem 28 *If we put* $(\underline{u}, \underline{v}) = (0,0)$ *and* $(\bar{u}, \bar{v}) = (L, L)$ *where* $L > 0$, *are pair of upper and lower solution of the system 27, such that* L *is big sufisamently, then the system* 27 *has maximal solution* (u^*, v^*) *and minimal solution* (u_*, v_*).

Case 2: If $T = \mathbb{Z}$, we have

$$\begin{cases} -\left(\varphi_p(u^\Delta)\right)^\Delta = u^{\lambda_1}(\sigma(t)) - u^{k_1}(\sigma(t)) + u^{\lambda_2}(\sigma(t))v^{k_2}(\sigma(t)) + h_1(t) \text{ in } [0,10]_{\mathbb{Z}}, \\ -\left(\varphi_q(u^\Delta)\right)^\Delta = u^{\lambda_3}(\sigma(t)) - u^{k_3}(\sigma(t)) + u^{\lambda_4}(\sigma(t))v^{k_4}(\sigma(t)) + h_2(t) \text{ in } [0,10]_{\mathbb{Z}}, \\ u(0) - u^\Delta(0) = \int_0^{12} g_1(s)u(s)\Delta s, u(12) + u^\Delta(11) = \int_0^{12} g_2(s)u(s)\Delta s, \\ v(0) - v^\Delta(0) = \int_0^{12} g_3(s)u(s)\Delta s, v(12) + v^\Delta(11) = \int_0^{12} g_2(s)v(s)\Delta s, \end{cases} \quad (29)$$

Then we have the result.

Theorem 29: *If we put* $(\underline{u},\underline{v}) = (0,0)$ *and* $(\bar{u},\bar{v}) = (L,L)$ *where* $L > 0$, *are pair of upper and lower solution of the system 27, such that* L *is big sufisamently, then the system 27 has maximal solution* (u^*,v^*) *and minimal solution* (u_*,v_*).

5.2 EXEMPLE 2

$$\begin{cases} -\left(\varphi_p(u^\Delta)\right)^\Delta = u^{\lambda_1}(\sigma(t)) - u^{k_1}(\sigma(t)) - u^{\lambda_2}(\sigma(t))v^{k_2}(\sigma(t)) + h_1(t) \text{ in } [0,10]_T, \\ -\left(\varphi_q(u^\Delta)\right)^\Delta = u^{\lambda_3}(\sigma(t)) - u^{k_3}(\sigma(t)) - u^{\lambda_4}(\sigma(t))v^{k_4}(\sigma(t)) + h_2(t) \text{ in } [0,10]_T, \\ u(0) - u^\Delta(0) = \int_0^{\sigma^2(10)} g_1(s)u(s)\Delta s, u\left(\sigma^2(10)\right) + u^\Delta\left(\sigma(10)\right) = \int_0^{\sigma^2(10)} g_2(s)u(s)\Delta s, \\ v(0) - v^\Delta(0) = \int_0^{\sigma^2(10)} g_3(s)u(s)\Delta s, v\left(\sigma^2(10)\right) + v^\Delta\left(\sigma(10)\right) = \int_0^{\sigma^2(10)} g_2(s)v(s)\Delta s, \end{cases} \quad (30)$$

Where:

$\lambda_i > 0$ and $k_i > 0$ for all $i = 1,2,3,4$ wich $k_1 > max(\lambda_1, \lambda_2 + k_2)$, and $k_3 > max(\lambda_3, \lambda_4 + k_4)$ and $h: [0,10]_T \to \mathbb{R}_+$are a contunious functions for $i = 1,2$ and $\int_0^{\sigma^2(10)} g_i(s)u(s)\Delta s < 1$ for all $i = 1,2,3,4$.

First of all it is easy to verify that the is increasing quasi-monotone system and the hypothesies $(H_1), (H_2)$ and (H_3) are holds.

We will study two cases.

Case 1: If $T = \mathbb{R}$, we have

$$\begin{cases} -\left(\varphi_p(u')\right)' = u^{\lambda_1}((t) - u^{k_1}((t) - u^{\lambda_2}((t)v^{k_2}((t) + h_1(t) \text{ in } [0,10], \\ -\left(\varphi_q(u')\right)' = u^{\lambda_3}((t) - u^{k_3}((t) - u^{\lambda_4}((t)v^{k_4}((t) + h_2(t) \text{ in } [0,10], \\ u(0) - u'(0) = \int_0^{10} g_1(s)u(s)ds, u(10) + u'(10) = \int_0^{10} g_2(s)u(s)\Delta s, \\ v(0) - v'(0) = \int_0^{10} g_3(s)u(s)ds, v(10) + v'(10) = \int_0^{10} g_2(s)v(s)ds, \end{cases} \quad (31)$$

Then we have the result

Theorem 30 If we put $(\underline{u}, \bar{u}) = (0, L)$ and $(\underline{v}, \bar{v}) = (L, L)$ where $L > 0$, are pair of upper and lower solution of the system 30, such that L is big sufisamently, then the system 30 has (u^*, v_*) and maximal-minimal solution and (u_*, v^*) minimalmaximal.

Case 2: If $T = \mathbb{Z}$, we have

$$\begin{cases} -\left(\varphi_p(u^\Delta)\right)^\Delta(t) = u^{\lambda_1}(t+1) - u^{k_1}(t+1) - u^{\lambda_2}(t+1)v^{k_2}(t+1) + h_1(t) \text{ in } [0,10]_{\mathbb{Z}}, \\ -\left(\varphi_q(u^\Delta)\right)^\Delta(t) = u^{\lambda_3}(t+1) - u^{k_3}(t+1) - u^{\lambda_4}(t+1)v^{k_4}(t+1) + h_2(t) \text{ in } [0,10]_{\mathbb{Z}}, \\ u(0) - u^\Delta(0) = \int_0^{12} g_1(s)u(s)\Delta s, u(12) + u^\Delta(11) = \int_0^{12} g_2(s)u(s)\Delta s, \\ v(0) - v^\Delta(0) = \int_0^{12} g_3(s)u(s)\Delta s, v(12) + v^\Delta(11) = \int_0^{12} g_2(s)v(s)\Delta s, \end{cases} \quad (32)$$

Then we have the result.

Theorem 31: *If we put $(\underline{u}, \bar{u}) = (0, L)$ and $(\underline{v}, \bar{v}) = (0, L)$ where $L > 0$, are pair of upper and lower solution of the system 31, such that L is big sufisamently, then the system 31 has (u^*, v_*) and maximal-minimal solution and (u_*, v^*) minimalmaximal.*

6 CONCLUSION

In a previous study, see[11] it was found that there exists the extrimals solutions of a quasilinear elliptic system with intergral boundary conditions in the continuous case, and from here we asked the following question using the upper and lower solutions method coupled with monotone iterative technique and we asked the following question: do we have the existence of extrimals solutions for certain classes of systems of differential equations with nonlocal boundary conditions in time scales by same method.

Differential equations in time scales have experienced considerable growth in recent years to explain several discrete phenomena, particularly in economics, biology and especially in computer science which uses discrete sets and therefore difference equations which are widely used to advance this science. It is very useful for describing seasonal phenomena, for example it could be for the study of a population of insects which after a certain time disappears, to reappear later after having been for a certain time in the form of a lava.

Through this work, we tried to make a qualitative shift with two dimensions: the first dimension is the transition from a nonlocal boundary conditions problem to certain classes of systems of differential equations with nonlocal boundary conditions and the second is the transition from the continuous state to the descret state. In this work we have shown the existence of the minimum and maximum solutions if the system of type 1 and of the maximum and minimum solutions if the system of type 2 on the other hand if the system is of type 3 we have shown only the existence of at least a near solution. Research is ongoing in the future to find complete solutions for the third type.

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