On the existence of positive solutions of some nonlocal elliptic problems involving fractional \( p \)-Laplacian operator

Sobre a existência de soluções positivas para alguns problemas ellípticos não locais que envolvem o operador fraccional \( p \)-Laplaciano

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ABSTRACT
Our aim in this paper is to analyze the existence of solutions to a nonlocal elliptic problem involving the fractional \( p \)-Laplacian operator. These operators are utilized to solve an equation defined within a bounded domain \( \Omega \) in \( \mathbb{R}^n \). The operator \( (-\Delta)^s_p \) is a fractional \( p \)-Laplacian, where \( 0 < s < 1 < p \) with the condition \( ps < n \) and \( h: \Omega \times \mathbb{R} \to \mathbb{R} \) is a non-negative function almost everywhere with respect to the variable \( x \), and \( \eta, \delta \) are real numbers. The article establishes two existence theorems for weak solutions using Tychonoff and Schauder fixed-point theorems. These theorems are formulated based on different hypotheses regarding the parameters \( \eta, \delta \) and \( h \).

Keywords: fractional \( p \)-Laplacian, fixed-point theorems, variational method, existence, weak solution.

RESUMO
Nosso objetivo neste artigo é analisar a existência de soluções para um problema elíptico não local envolvendo o operador \( p \)-Laplaciano fracionário. Esses operadores são utilizados para resolver uma equação definida em um domínio delimitado \( \Omega \) em \( \mathbb{R}^n \). O operador \( (-\Delta)^s_p \) é um \( p \)-Laplaciano fracionário, em que \( 0 < s < 1 < p \) com a condição \( ps < n \) e \( h: \Omega \times \mathbb{R} \to \mathbb{R} \) é uma função não negativa em quase todos os lugares com relação à variável \( x \), e \( \eta, \delta \) são números reais. O
The objective of this paper is to demonstrate the existence of weak solutions under various assumptions about the real numbers $\delta, \eta$ and function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ for the following non-local fractional $p$–Laplacian operator,

\[
\begin{cases}
(-\Delta)^s_p w = \frac{(h(x,w))^{\eta}}{(\int_{\Omega} h(x,w)dx)^{\eta}}, & w > 0, \text{ in } \Omega \\
w = 0 & \text{on } \mathbb{R}^n \setminus \Omega
\end{cases}
\]  \hspace{1cm} (1.1)

The $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain, $0 < s < 1 < p$ satisfy that $ps < n$ and

\[
(-\Delta)^s_p w(x) : \text{P.V.} \int_{\mathbb{R}^n} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{n+ps}} dy,
\]

is the fractional $p$–Laplacian operator, where P.V stands for Cauchy’s principal value.

We introduce two sets of hypotheses.

First, let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $\nu \mapsto h(x, \nu)$ is continuous for a.e. $x \in \Omega$, for every $\nu \in \mathbb{R}$, $x \mapsto h(x, \nu)$ is measurable and for some function $f \in L^1(\Omega)$

\[
\text{for all } \nu \in \mathbb{R}, \text{ a.e. } x \in \Omega, 0 < h(x, \nu) \leq f(x) \tag{1.2}
\]

and $\eta, \delta$ satisfies one of following assumptions:

\[
\delta \leq \frac{p-1}{p} < \eta \text{ with } f(x) < 1 \text{ a.e. } x \in \Omega \tag{1.3}
\]
\[ \delta \leq \eta \text{ and } \eta \in \left[0, \frac{p-1}{p}\right]; \quad (1.4) \]

In the second set of hypotheses, we consider \( h : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) a measurable function for all \( v \in \mathbb{R} \) and

\[ h(x, v) > 0, \forall v \in \mathbb{R}, \text{ a.e. } x \in \Omega \quad (1.5) \]

\[ g(., 0) \in L^1(\Omega), |h(x, v_1) - h(x, v_2)| \leq H_\tau(x)|v_1 - v_2|\tau, \forall v_1, v_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega \text{ such that } (1.6) \]

\[ \tau \in (0, p), H_\tau \in L^p(\Omega), \text{ and for } \tau = p, H_p \in L^\infty(\Omega), 0 < \delta \leq \eta \leq \frac{p-1}{p} \quad (1.7) \]

Therefore, the two existence theorems for weak solutions used to address the problem (1.1) are proven. The evidence is based on the two sets of hypotheses specified for \( h, \eta \) and \( \delta \), utilizing the Tychonoff and Schauder fixed-point theorems, respectively.

For the local \( p \)–Laplacian operator \((s = 1)\), we note the following problem:

\[-\Delta_p w = \frac{(h_{xw})^\eta}{(f_{xw} + k_{xw})^\delta}, w > 0, \text{ in } \Omega, w = 0 \text{ on } \partial \Omega \quad (1.8)\]

The problem described in (1.8) includes measurable functions \( h \) and \( f \), with real parameters \( \delta \) and \( \eta \). Several studies have been dedicated to address problems of this form. This semi-linear case with \( p = 2 \) and \( h(x, w) = f(x, w) = T(w) \) depends only on the variable of \( w \) and was addressed in [2]. The authors utilized a fixed-point theorem within a cone to prove that at least one non-negative solution does exist for problem (1.8). This assertion holds under the premise that \( T \) is a continuous nonnegative non-decreasing function satisfying a conditional nature. Reference [22] establishes the existence of solutions that are unique for problem (1.8) when \( p = 2 \). These results are deduced under specific conditions imposed on the functions \( h, f \) and the constants \( \eta \) and \( \delta \) (For the specific type of nonlocal problems described refer to [18, 6, 24, 25]). In [19], the author provided
sufficient conditions on the function $h$, as well as on $\delta$ and $\eta$, to establish the existence of weak non-trivial nonnegative solutions. This was accomplished using the Tychonoff and Schauder fixed-point theorems. The conditions and theorems were applied to a class of problems characterized by a specific form

$$
-\text{div} \left( \left( a_{ij}(x) \right) \nabla w \right) = \frac{(h(x,w))^\eta}{(\int_\Omega h(x,w)dx)^\sigma}, \text{ in } \Omega, w=0 \text{ on } \partial \Omega, w > 0 \text{ in } \Omega
$$

(1.9)

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain and for a.e. in $\Omega, \left( a_{ij}(x) \right)$ be an $n \times n$ matrix function. Further explore studies like [5, 21, 16] for a comprehensive understanding of various non-local elliptic problems, particularly when $p > 1$. The literature surrounding non-local operators and their diverse applications is extensive including [4, 8, 9, 10, 11, 12, 15, 23].

In this paper we compile essential preliminaries in Section 2, establishing the functional setting for our problems. This section introduces crucial concepts like solutions and relevant lemmas, laying the groundwork for the paper’s coherence and development. In section 3, by using Tychonoff fixed-point theorem, we show the existence of nonnegative solution of problem (1.1) under the assumption (1.2) and one of (1.4) or (1.3). Lastly and through applying Schauder fixed-point theorem, we show the existence of weak solution to (1.1) under mentioned assumptions on $h, \eta$ and $\delta$.

2 PRELIMINARIES

This section outlines key results on Sobolev space and introduce essential tools for the main arguments. Assume that $0 < s < 1 < p$ and $\Omega \subset \mathbb{R}^n$ open-bounded set. We denote $W^{s,p}(\Omega)$ as the space of functions $v \in L^p(\Omega)$ satisfying

$$
\int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+ps}} dxdy < +\infty
$$

Evidently, $W^{s,p}(\Omega)$ forms a Banach space endowed with the norm

$$
\|v\|_{W^{s,p}(\Omega)} = \left( \|v\|_{L^p(\Omega)}^p + \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}}
$$

Similarly, we define the space
$W_0^{s,p}(\Omega) := \{ v \in W^{s,p}(\mathbb{R}^n) : v = 0 \text{ in } \mathbb{R}^n \setminus \Omega \}.$

Then, $W_0^{s,p}(\Omega)$ endowed with the norm

$$\|v\|_{W_0^{s,p}(\Omega)} := \left( \int_{D\Omega} \frac{|v(x) - v(y)|^p}{|x-y|^{n+ps}} \, dx \, dy \right)^{\frac{1}{p}},$$

where $D\Omega := (\Omega \times \mathbb{R}^n) \cup (\Omega \times \mathbb{R}^n) = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (C\Omega \times C\Omega)$.

Then

$$W^{-s,p}(\Omega) := \left( W_0^{s,p}(\Omega) \right)^{\prime}, \quad p^{\prime} := \frac{p}{p-1}$$

here, the last set refers to the reflexive Banach space, and its pairing with $W_0^{s,p}(\Omega)$ is denoted by $(\cdot, \cdot)$.

For detailed properties and further information regarding fractional Sobolev spaces, refer to [1, 3, 13]. Further recalling the Sobolev embedding theorem [13].

**Theorem 2.1.** Let $0 < s < 1 < p$ and $ps < n$. Then there exists a positive constant $C \equiv C(n,s,p)$ such that for every $v \in C_0^\infty(\mathbb{R}^n)$,

$$C\|v\|_{L^{p^*}(\mathbb{R}^n)}^p \leq \int_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^p}{|x-y|^{n+ps}} \, dx \, dy,$$

where $p^* := \frac{pn}{n-ps}$ the fractional critical exponent.

For $0 < s < 1 < p < \infty$, and $\Omega \subset \mathbb{R}^n$ is an open bounded domain. Consider the following quasi-linear problem,

$$\begin{cases}
(-\Delta)^s v = g, \text{ in } \Omega \\
 w = 0 \quad \text{ on } \mathbb{R}^n \setminus \Omega
\end{cases} \quad (2.1)$$

For $g \in W_0^{s,p}(\Omega)$, we assume that $v \in W_0^{s,p}(\Omega)$ is a weak solution to (2.1) if

$$\frac{1}{2} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(\psi(x) - \psi(y))}{|x-y|^{n+ps}} \, dx \, dy = \int_\Omega g \varphi \, dx, \ \forall \varphi \in W_0^{s,p}(\Omega),$$
Proposition 2.2. [17]. Let $0 < s < 1 < p < \infty$. Then, we have that

(i) $(-\Delta)^s_p : W^s_0(\Omega) \to W^{-s,p}(\Omega)$ is continuous, bounded, strictly monotone and coercive.

(ii) $[(-\Delta)^s_p]^{-1} : W^{-s,p}(\Omega) \to W^s_0(\Omega)$ is lipshitz continuous if $p \geq 2$, and locally lipshitz continuous if $p \in (1, 2)$.

(iii) The composed operator $W^{-s,p}(\Omega) \to W^s_0(\Omega) \to L^\sigma(\Omega)$ is compact if $1 \leq \sigma < p^*_s$.

Proposition 2.3. [17]. Let $p \in (1, \infty)$ and $s \in (0,1)$. Then for every $g \in W^{-s,p}(\Omega)$, the Dirichlet problem (2.1) has a unique weak solution $v \in W^s_0(\Omega)$.

Moreover,

$$\|v\|_{W^s_0(\Omega)} \leq \|g\|_{W^{-s,p}(\Omega)}^{1/p=1}$$

(2.2)

We introduce the following elementary algebraic inequality to be used later.

Lemma 2.4. [20] Let $p > 1$. There exists a positive constant $C = C(p) > 0$ such that, for every $v_1, v_2 \in \mathbb{R}^n$

$$\langle |v_1 - v_2|^{p-2}(v_1 - v_2), (v_1 - v_2) \rangle \geq C \frac{|v_1 - v_2|^2}{(|v_2|^2 + |v_1|^2)^{2-p}}$$

(2.3)

Now, let us introduce our definition of solution to the problem (1.1).

Definition 2.5. A function $w \in W^s_0(\Omega)$ is said to be a weak solution to problem (1.1), if $w > 0$ in $\Omega$, $w = 0$ in $\mathbb{R}^n \setminus \Omega$ and for every $\psi \in W^s_0(\Omega)$, we have

$$\int_\Omega \psi(-\Delta)^s_p w dx = \frac{1}{(\int_\Omega h(x,w) dx)^s} \int_\Omega \left(h(x,w)\right)^\eta \psi dx$$

(2.4)

Noting that we will use the following notations throughout the paper. The expression H.P.I refers to Holder and Poincaré Inequalities, and the term a.e. in $\Omega$ is abbreviated almost everywhere in $\Omega$. If $\tilde{B}_R$ is a subset of a space $Y$, where $\tilde{B}_R := \{a \in Y; \|a\|_Y \leq R\}$ the closed ball at 0 with radius $R > 0$. 

3 FIRST EXISTENCE THEOREM

We start by presenting the first theorem.

**Theorem 3.1.** Let $0 < s < 1 < p$ satisfy $p < ns$. Assume that (1.2) and either (1.4) or (1.3) hold, the problem (1.1) possesses a nonnegative weak solution $w \in W_0^{s,p} (\Omega)$.

**Proof.** Three steps will be used to generate evidence.

**Step 1:** The following problem is considered:

$$\begin{cases}
(-\Delta)^s w = \frac{(h(x,\xi))^\eta}{(\int_\Omega h(x,\xi) dx)^\frac{s}{p}}, w > 0, \text{in } \Omega \\
w = 0 \quad \text{on } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

for all $\xi \in W_0^{s,p} (\Omega)$. To establish a solution $w \in W_0^{s,p} (\Omega)$, it is essential to consider the following:

$$\Psi (\xi) = \frac{1}{(\int_\Omega h(x,\xi) dx)^\frac{s}{p}} \left( \int_\Omega (h(x,\xi))^\eta \right) \in W^{-s,p} (\Omega)$$

(3.2)

To do this, choosing $\psi \in W_0^{s,p} (\Omega)$ as a test function in (3.1), we obtain that:

$$|\langle \Psi (\xi), \psi \rangle| \leq \frac{1}{(\int_\Omega h(x,\xi) dx)^\frac{s}{p}} \left( \int_\Omega (h(x,\xi))^\eta |\psi| dx \right)$$

by applying H.P.I, we have:

$$|\langle \Psi (\xi), \psi \rangle| \leq \frac{1}{(\int_\Omega h(x,\xi) dx)^\frac{s}{p}} \left( \int_\Omega (h(x,\xi))^\eta |\psi|^p dx \right)^{\frac{1}{p}} \leq C \|\psi\|_{W_0^{s,p} (\Omega)} \left( \int_\Omega (h(x,\xi))^\eta dx \right)^{\frac{1}{p}}$$

First, supposing that (1.3) holds, we obtain:

$$|\langle \Psi (\xi), \psi \rangle| \leq C \|\psi\|_{W_0^{s,p} (\Omega)} \left( \int_\Omega (h(x,\xi))^\eta dx \right)^{\frac{1}{p}} = C \|\psi\|_{W_0^{s,p} (\Omega)} \left( \int_\Omega (h(x,\xi) dx \right)^{\frac{1}{p}}$$
\[
\| f \|_{L^1(\Omega)} \| \psi \|_{W_0^{s,p}(\Omega)} \leq C \| f \|_{L^1(\Omega)} \| \psi \|_{W_0^{s,p}(\Omega)}
\] (3.3)

Now, assuming that (1.4) holds, it follows that:

if \( \eta \geq \delta \) and \( 0 < \eta < \frac{p-1}{p} \), we obtain

\[
|\langle \Psi(\xi), \psi \rangle| \leq C|\Omega| \frac{p}{p-\eta} \| \psi \|_{W_0^{s,p}(\Omega)} \left( \int_{\Omega} h(x,\xi) dx \right)^{\eta-\delta} \leq C|\Omega| \frac{p}{p-\eta} \| f \|_{L^1(\Omega)} \| \psi \|_{W_0^{s,p}(\Omega)}
\] (3.4)

if \( \eta \geq \delta \) and \( \eta = 0 \), we have

\[
|\langle \Psi(\xi), \psi \rangle| \leq C|\Omega| \frac{p}{p}\| \psi \|_{W_0^{s,p}(\Omega)} \left( \frac{1}{\int_{\Omega} h(x,\xi) dx} \right)^{\delta} \leq C|\Omega| \frac{p-1}{p-\delta} \| f \|_{L^1(\Omega)} \| \psi \|_{W_0^{s,p}(\Omega)}
\] (3.5)

if \( \eta \geq \delta \) and \( \eta = \frac{p-1}{p} \), we get

\[
|\langle \Psi(\xi), \psi \rangle| \leq C \| \psi \|_{W_0^{s,p}(\Omega)} \left( \frac{1}{\int_{\Omega} h(x,\xi) dx} \right)^{\frac{1}{p}} \leq C \| f \|_{L^1(\Omega)} \| \psi \|_{W_0^{s,p}(\Omega)}
\] (3.6)

Hence, combining (3.4), (3.5), (3.6) and (3.3) as follows, we can apply the Proposition 2.3 to show that for each \( \xi \in W_0^{s,p}(\Omega) \), it is evident the existence of a unique weak solution \( w \in W_0^{s,p}(\Omega) \) to (3.1) given by

\[
\langle (-\Delta)^{\frac{s}{p}} w, \psi \rangle = \int_{\Omega} \frac{(h(x,\xi))^\eta \psi(x) dx}{\left( \int_{\Omega} h(x,\xi) dx \right)^{\delta}}
\]

for all \( \psi \in W_0^{s,p}(\Omega) \).

Therefore, we define the following operator,

\[
G : W_0^{s,p}(\Omega) \to W_0^{s,p}(\Omega)
\]
\[
\xi \mapsto w = G(\xi)
\]

is well-defined.
Step 2: We will now prove that $G(\bar{B}_R) \subset \bar{B}_R$ where $\bar{B}_R$ in $W_0^{s,p}(\Omega)$. To establish this, it is adequate to say that if $v$ solves (3.1), then

$$\|v\|_{W_0^{s,p}(\Omega)} \leq R$$

where specific value of $R = R(|\Omega|, \|f\|_{L^1(\Omega)}, p, \eta, \delta) > 0$ will be determined later.

Use of $w$ as a test function in (3.1), yields:

$$\langle (-\Delta)^s w, w \rangle = \frac{\int_{\Omega} (h(x, \xi))^\eta w(x) dx}{(\int_{\Omega} h(x, \xi) dx)^{\delta}}$$

which, by using H.S.I, we obtain

$$\|w\|_{W_0^{s,p}(\Omega)}^p \leq \frac{C \|w\|_{W_0^{s,p}(\Omega)}^{\frac{p}{p-1}}}{(\int_{\Omega} h(x, \xi) dx)^{\delta}} \left( \int_{\Omega} (h(x, \xi))^{\eta p'} dx \right)^{\frac{1}{p}}$$

so that

$$\|w\|_{W_0^{s,p}(\Omega)}^{p-1} \leq \frac{C}{(\int_{\Omega} h(x, \xi) dx)^{\delta}} \left( \int_{\Omega} (h(x, \xi))^{\eta p'} dx \right)^{\frac{1}{p}}.$$

(3.7)

Now, by using (1.4), we distinguish three cases similar in Step 1 using H.S.I, we deduce from (3.7), for $\eta \geq \delta$ and $\eta = \frac{p-1}{p}$,

$$\|w\|_{W_0^{s,p}(\Omega)}^{p-1} \leq C \left( \int_{\Omega} h(x, \xi) dx \right)^{\frac{1}{p}} \left( \int_{\Omega} (h(x, \xi))^{\eta p'} dx \right)^{\frac{1}{p}} \leq C \|f\|_{L^1(\Omega)}^{p-1}$$

which implies

$$\|w\|_{W_0^{s,p}(\Omega)} \leq C^{p-1} \|f\|_{L^1(\Omega)} \frac{1}{p-\delta} \quad (3.8)$$
If $\delta \leq \eta = 0$,

$$\|w\|_{W_0^{p,1}(\Omega)}^{p-1} \leq C \frac{|\Omega|^\frac{1}{p}}{(\int_{\Omega} h(x, \xi) \, dx)^\delta} \leq C |\Omega|^\frac{1}{p} \|f\|_{L^1(\Omega)}^{1-\delta}$$

which leads to

$$\|w\|_{W_0^{p,1}(\Omega)} \leq C \frac{1}{p-1} |\Omega|^\frac{1}{p} \|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}$$  \hspace{1cm} (3.9)$$

if $\delta \leq \eta$ and $\eta \in \left(0, \frac{p-1}{p}\right)$,

$$\|w\|_{W_0^{p,1}(\Omega)} \leq C \frac{(\int_{\Omega} (h(x, \xi))^{p'} \, dx)^\frac{1}{p}}{(\int_{\Omega} h(x, \xi) \, dx)^\delta} \leq C |\Omega|^\frac{1}{p-1} \|f\|_{L^1(\Omega)}^{\frac{1}{p-1} - \eta} \leq C |\Omega|^\frac{1}{p-1} \|f\|_{L^1(\Omega)}^{\frac{1}{p-1} - \eta - \delta}$$

which implies

$$\|w\|_{W_0^{p,1}(\Omega)} \leq C \frac{1}{p-1} |\Omega|^\frac{1}{p-1} \|f\|_{L^1(\Omega)}^{\frac{1}{p-1} - \eta - \delta}$$  \hspace{1cm} (3.10)$$

Moreover, when (1.3) is hold, we have

$$\|w\|_{W_0^{p,1}(\Omega)} \leq C \frac{1}{p-1} |\Omega|^\frac{1}{p-1} \|f\|_{L^1(\Omega)}^{\frac{1}{p-1} - \eta - \delta}$$

which implies

$$\|w\|_{W_0^{p,1}(\Omega)} \leq C \frac{1}{p-1} \|f\|_{L^1(\Omega)}^{\frac{1}{p-1} - \eta - \delta}$$  \hspace{1cm} (3.11)$$
Hence, bases on (3.8)-(3.11), we conclude that if (1.2) and one of (1.4)-(1.3) hold, there exists a constant $R > 0$ such that $G(B_R) \subset B_R$.

**Step 3:** We will check here that $G : B_R \rightarrow B_R$ is weakly continuous. Let $\{\xi_k\}_{k \in I}$ be a generalized sequence in $B_R$ such that $\xi_k \rightharpoonup \xi$ weakly in $W^{s,p}_0(\Omega)$, we consider $G(\xi_k) = w_k$ and $G(\xi) = w$. We shall aim to prove that $\{w_k\}_{k \in I}$ converges to $w$ weakly in $B_R$. Since $B_R$ is weakly compact in $W^{s,p}_0(\Omega)$, it is sufficient to show that $\{w_k\}_{k \in I}$ has $w$ as unique weak limit for that topology. Assuming the contrary,

$$w_k \rightharpoonup \hat{w} \text{ weakly in } W^{s,p}_0(\Omega). \quad (3.12)$$

Moreover, we know that $W^{s,p}_0(\Omega) \rightarrow L^p(\Omega)$ is compact (by Proposition 2.2). Consequently, there exists a subsequence of $\{\xi_k\}_{k \in I}$ such that

$$\xi_k(x) \rightarrow \xi(x) \quad \text{a.e. in } \Omega. \quad (3.13)$$

As both $w_k$ and $w$ are nonnegative solutions to (3.1), choosing $\psi = (w_k - w)$ as a test function in (3.1). Subtracting the two equations, we arrive at

$$\langle (-\Delta)^s_p w_k - (-\Delta)^s_p w, \psi \rangle = \frac{\int_\Omega (h(x,\xi_k))^{\gamma} \psi \, dx}{(\int_\Omega h(x,\xi_k) \, dx)^{\sigma}} - \frac{\int_\Omega (h(x,\xi))^{\gamma} \psi \, dx}{(\int_\Omega h(x,\xi) \, dx)^{\sigma}} \quad (3.14)$$

Hence, by utilizing Lemma 2.3, for $p \geq 2$ it holds that:

$$\langle (-\Delta)^s_p w_k - (-\Delta)^s_p w, w_k - w \rangle \geq c\|w_k - w\|_{W^{s,p}_0(\Omega)}^p$$

Combining with (3.14) and (3.15), we conclude that:

$$c\|w_k - w\|_{W^{s,p}_0(\Omega)}^p \leq \frac{1}{(\int_\Omega h(x,\xi_k) \, dx)^{\delta}} \int_\Omega \left\{ (h(x,\xi_k))^{\gamma} - (h(x,\xi))^{\gamma} \right\} (w_k - w) \, dx$$

$$+ \left\{ \frac{1}{(\int_\Omega h(x,\xi_k) \, dx)^{\delta}} - \frac{1}{(\int_\Omega h(x,\xi) \, dx)^{\delta}} \right\} \int_\Omega (h(x,\xi))^{\gamma} (w_k - w) \, dx. \quad (3.16)$$

Furthermore, utilizing (1.2), we have
\[ \int_{\Omega} h(x, \xi_k) \, dx > 0 \quad \text{and} \quad \int_{\Omega} h(x, \xi) \, dx > 0, \text{ for every } j \in \mathbb{N}. \] (3.17)

So,

\[ (\int_{\Omega} h(x, \xi_k) \, dx)^{\delta} \rightarrow (\int_{\Omega} h(x, \xi) \, dx)^{\delta} \text{ strongly in } \mathbb{R}, \] (3.18)

And

\[ (h(x, \xi_k))^{\eta} \rightarrow (h(x, \xi))^{\eta} \text{ strongly in } L^{\frac{p}{p-1}}(\Omega), \] (3.19)

using (1.2), (3.13) and by the dominated convergence theorem, we get (3.18)-(3.19).

Observe that \( w_k \in \tilde{B}_R \). Using (3.18)-(3.19) and passing to the limit \( k \rightarrow \infty \) in (3.16), we get

\[ w_k \rightarrow w \quad \text{strongly in } W^{s,p}_0(\Omega). \]

Considering (3.12), we have \( w = \tilde{w} \). Thus, \( w \) is the unique weak limit point of \( \{w_k\}_{k \in I} \), and therefore,

\[ w_k = G(\xi_k) \rightharpoonup w = G(\xi) \quad \text{weakly in } W^{s,p}_0(\Omega). \]

Now, we aim to show that if \( h(x, \nu) \geq 0 \) for all \( \nu \in (-\infty, 0) \) and a.e. \( x \in \Omega \), any solution \( w \) of the Problem (3.1) satisfies \( w \geq 0 \) a.e. in \( \Omega \). We select \( w^- \) as a test function in (3.1), resulting in:

\[ \langle (-\Delta)^{\frac{\delta}{2}} w, w^- \rangle = \frac{\int_{\Omega} (h(x, \xi))^{\eta} w^- \, dx}{(\int_{\Omega} h(x, \xi) \, dx)^{\delta}} \leq 0. \]

This leads to:

\[ \langle (-\Delta)^{\frac{\delta}{2}} w, w^- \rangle \leq 0 \]

Thus,
\[ \|w^\pm\|_{W_0^{s,p}(\Omega)} \leq 0 \]

Which leads to \( w^\pm = 0 \) a.e. in \( \Omega \). Since \( w = 0 \) in \( \mathbb{R}^n \setminus \Omega \).

The case \( p \in (1,2) \) is handled with similar arguments, so we omit the details. Finally, the operator \( G \) has a fixed-point \( w \in B_R \) such that \( w = G(w) \), constituting a weak solution to (1.1), by applying the Tychonoff fixed-point theorem.

### 4 THE SECOND EXISTENCE THEOREM

Using the Schauder fixed-point theorem to show that the problem (1.1) admits a nonnegative solution.

**Theorem 4.1.** Let \( 0 < s < 1 \) and \( 1 < p < ns \). Assume that (1.5)-(1.7) hold, the problem (1.1) possesses a nonnegative weak solution \( w \in W_0^{s,p}(\Omega) \).

**Proof.** Three steps will be used to generate evidence.

**Step 1:** The following problem is considered:

\[
\begin{cases}
(-\Delta)^s_p w = \frac{(h(x,\xi))^{s\tau}}{(\int_\Omega h(x,\xi)dx)}^\frac{s}{\tau}, & w > 0, \text{ in } \Omega \\
w = 0, & \text{ on } \mathbb{R}^n \setminus \Omega,
\end{cases}
\tag{4.1}
\]

We will check here that

\[ Y(\xi) = \frac{1}{(\int_\Omega h(x,\xi)dx)}^\frac{s}{\tau} (h(x,\xi))^\frac{s\tau}{\tau} \in W^{-s,p}(\Omega) \text{ for all } \xi \in W_0^{s,p}(\Omega), \tag{4.2} \]

following the approach in the proof of the previous result, noting

- if \( h \) satisfies (1.6) with \( \tau \in (0, p) \), \( H_p \in L^{\infty}(\Omega) \) and \( H_\tau \in L^{\frac{p}{p-\tau}}(\Omega) \) for \( 0 < \tau < p \), and using H.P.I, we deduce that \( h \in (,\xi) \in L^1(\Omega) \) and \( (h \in (,\xi))^\frac{s\tau}{\tau} \in L^p(\Omega) \). We obtain:

\[ Y(\xi) \in W^{-s,p}(\Omega). \]

- Note that \( h(x,\xi) > 0 \) for all \( \xi \in \mathbb{R} \) and a.e. \( x \in \Omega \). We conclude that \( w > 0 \) in \( \Omega \).
Then, we can apply the Proposition 2.3 to show that for each $\xi \in W_0^{s,p}(\Omega)$, there exists a unique weak solution $w \in W_0^{s,p}(\Omega)$ to Problem (4.1).

Step 2: We consider the operator $N$ defined by:

$$N : L^p(\Omega) \to W_0^{s,p}(\Omega)$$

$$\xi \mapsto w = N(\xi),$$

Setting $\bar{B}_R \subset L^p(\Omega)$.

Lemma 4.2. There exist $R > 0$ such that:

1) There exists $R > 0$ such that $N(\bar{B}_R) \subset \bar{B}_R$,

2) $N : \bar{B}_R \to \bar{B}_R$ is continuous,

3) $N(\bar{B}_R)$ is relatively compact in $L^p(\Omega)$.

Proof. 1) Choosing $w$ as a test function in (4.1) and applying H.P.I, leads to

$$\|w\|_{L^p(\Omega)}^p \leq C \left( \int_{\Omega} h(x, \xi) \eta^p \frac{dx}{\int_{\Omega} h(x, \xi) dx} \right)^\frac{1}{p} \|w\|_{L^p(\Omega)}.$$ 

that can be written as

$$\|w\|_{L^p(\Omega)}^{p-1} \leq C \left( \frac{\int_{\Omega} h(x, \xi) \eta^p \frac{dx}{\int_{\Omega} h(x, \xi) dx} \right)^\frac{1}{p}.$$ 

which, by using (1.5), (1.6) and H.P.I, we get

- if $0 < \delta < \eta$ and $\eta = \frac{p-1}{p}$,

$$\|w\|_{L^p(\Omega)}^{p-1} \leq C \left( \frac{\int_{\Omega} h(x, \xi)\eta^p \frac{dx}{\int_{\Omega} h(x, \xi) dx} \right)^\frac{1}{p} \leq C \left( \int_{\Omega} |H_\tau(x) \xi|^p + h(x, 0) \right)^{\frac{1}{p^\tau}} \leq C \left( \left( \int_{\Omega} |H_\tau(x) \xi|^p + h(x, 0) \right)^{\frac{1}{p^\tau}} \right)^{\frac{1}{p}}.$$
\[ \leq C \left( \|H\|_{L^p(\Omega)} \frac{1}{\tau^{\frac{\eta^\delta}{p}}} \|\xi\|_{L^p(\Omega)} + \|h(x,0)\|_{L^1(\Omega)} \right). \]

which leads to

\[ \|w\|_{L^p(\Omega)} \leq \begin{cases} C_1 \left( \|H\|_{L^p(\Omega)} \frac{1}{\tau^{\frac{\eta^\delta}{p}}} \|\xi\|_{L^p(\Omega)} + \|h(.0)\|_{L^1(\Omega)} \right) & \text{if } 0 < \tau < p, \\
C_2 \left( \|H\|_{L^p(\Omega)} \frac{1}{\tau^{\frac{\eta^\delta}{p}}} \|\xi\|_{L^p(\Omega)} + \|h(.0)\|_{L^1(\Omega)} \right) & \text{if } \tau = p. \end{cases} \]

Therefore, since \( 0 < \tau \left( \frac{1}{p} - \frac{\delta}{p-1} \right) < 1 \) for every \( \tau \in 0, p \) there exists \( R_1 > 0 \) depend \( p, \tau, \eta \) and \( \delta \) such that

\[ \|w\|_{L^p(\Omega)} \leq R_1; \quad (4.4) \]

- if \( 0 < \delta < \eta < \frac{p-1}{p} \), we arrive at

\[ \|w\|_{L^p(\Omega)} \leq C|\Omega|^{-\eta^\delta} \left( \int_{\Omega} h(x,\xi)dx \right)^{\eta^\delta} \leq C|\Omega|^{-\eta^\delta} \left( \int_{\Omega} (H(x)|\xi|^\tau + h(x,0))dx \right)^{\eta^\delta} \]

\[ \leq C|\Omega|^{-\eta^\delta} \left( \|H\|_{L^p(\Omega)} \frac{1}{\tau^{\frac{\eta^\delta}{p}}} \|\xi\|_{L^p(\Omega)} + \|h(.0)\|_{L^1(\Omega)} \right)^{\eta^\delta} \]

\[ \leq \begin{cases} C|\Omega|^{-\eta^\delta} \left( \|H\|_{L^p(\Omega)} \frac{1}{\tau^{\frac{\eta^\delta}{p}}} \|\xi\|_{L^p(\Omega)} + \|h(.0)\|_{L^1(\Omega)} \right) & \text{if } 0 < \tau < p, \\
C|\Omega|^{-\eta^\delta} \left( \|H\|_{L^p(\Omega)} \frac{1}{\tau^{\frac{\eta^\delta}{p}}} \|\xi\|_{L^p(\Omega)} + \|h(.0)\|_{L^1(\Omega)} \right) & \text{if } \tau = p. \end{cases} \]

Therefore,

\[ \|w\|_{L^p(\Omega)} \leq \begin{cases} C_2|\Omega|^{-\eta^\delta} \left( \|H\|_{L^p(\Omega)} \frac{1}{\tau^{\frac{\eta^\delta}{p}}} \|\xi\|_{L^p(\Omega)} + \|h(.0)\|_{L^1(\Omega)} \right) & \text{if } 0 < \tau < p, \\
C_2|\Omega|^{-\eta^\delta} \left( \|H\|_{L^p(\Omega)} \frac{1}{\tau^{\frac{\eta^\delta}{p}}} \|\xi\|_{L^p(\Omega)} + \|h(.0)\|_{L^1(\Omega)} \right) & \text{if } \tau = p. \end{cases} \]
Since, $0 < \frac{\tau(\eta - \delta)}{p - 1} < 1$, we obtain for some positive constant $R_2 > 0$ depend $p, \tau, \eta$ and $\delta$ such that

$$\|w\|_{L^p(\Omega)} \leq R_2; \quad (4.5)$$

- if $0 < \delta \leq \frac{p - 1}{p}$ and $\eta = \delta$, we arrive at

$$\|w\|_{L^p(\Omega)} \leq C_3 |\Omega|^\frac{1}{p} \frac{\delta}{p - 1}. \quad (4.6)$$

We deduce from (4.4)-(4.6) that for all $\tau \in 0, p$, there exists a constant $R$ such that $N(\bar{B}_R) \subset \bar{B}_R$.

2) Let $\{\xi_i\}_i \subset L^p(\Omega)$ such that

$$\xi_i \to \xi \text{ in } L^p(\Omega). \quad (4.7)$$

Using (4.1) for both $w_i$ and $w$ with $(w_i - w)$ as test function, and subtracting the two identities from each other. Similar to proof Theorem 3.1 using the same computation $p \geq 2$, we obtain:

$$\|w_i - w\| \leq \frac{1}{(\int_{\Omega} h(x, \xi_i) dx)^\frac{1}{p}} \left\{ \frac{1}{(\int_{\Omega} h(x, \xi) dx)^\frac{1}{p}} \right\} \int_{\Omega} (h(x, \xi_i))^p (w_i - w) dx$$

$$= \frac{1}{(\int_{\Omega} h(x, \xi_i) dx)^\frac{1}{p}} \left\{ \frac{1}{(\int_{\Omega} h(x, \xi) dx)^\frac{1}{p}} - \frac{1}{(\int_{\Omega} h(x, \xi) dx)^\frac{1}{p}} \right\} \int_{\Omega} (h(x, \xi))^p (w_i - w) dx. \quad (4.8)$$

Now, we will estimate $I_1$ and using H.S.I, we get

$$|I_1| \leq \int_{\Omega} |(h(x, \xi_i))^p - (h(x, \xi))^p| |w_i - w| dx \leq \int_{\Omega} |(h(x, \xi_i) - h(x, \xi))^p| |w_i - w| dx$$

$$\leq \left( \int_{\Omega} |h(x, \xi_i) - h(x, \xi)|^{p'} dx \right)^{\frac{1}{p'}} \|w_i - w\|_{L^p(\Omega)}. \quad (4.9)$$

Then, using (1.6), H.S.I, we destinguish two cases from (4.9)
for $\eta = \frac{p-1}{p}$

\[
|I_1| \leq \left( \int_{\Omega} |h(x, \xi) - h(x, \xi)|^{\eta p} \, dx \right)^{\frac{1}{p}} \|w_i - w\|_{L^P(\Omega)}
\]

\[
\leq \begin{cases} 
\left( \int_{\Omega} |H_{\tau}(x)| \xi_i - \xi |^p \, dx \right)^{\frac{1}{p}} \|w_i - w\|_{L^P(\Omega)}, & \text{if } \tau \in (0, p), \\
\left( \int_{\Omega} |H_p(x)| \xi_i - \xi |^p \, dx \right)^{\frac{1}{p}} \|w_i - w\|_{L^P(\Omega)}, & \text{if } \tau = p,
\end{cases}
\]

\[
(4.10)
\]

and if $\eta \in (0, \frac{p-1}{p})$,

\[
|I_1| \leq |\Omega| \left( \int_{\Omega} |h(x, \xi) - h(x, \xi)| \, dx \right)^{\eta} \|w_i - w\|_{L^p(\Omega)}
\]

\[
\leq |\Omega| \left( \int_{\Omega} H_{\tau}(x) |\xi_i - \xi |^p \, dx \right)^{\eta} \|w_i - w\|_{L^p(\Omega)}
\]

\[
(4.11)
\]

In contrast,

\[
\left| \int_{\Omega} h(x, \xi_i) \, dx - \int_{\Omega} h(x, \xi) \, dx \right| \leq \int_{\Omega} |h(x, \xi) - h(x, \xi)| \, dx
\]

\[
\leq \int_{\Omega} |H_{\tau}(x)| |\xi_i - \xi |^{p} \, dx
\]
\[
\|w\|_{L^p(\Omega)} \leq \left\{
\begin{array}{ll}
\left( C \left( |\Omega|^{\frac{1}{p}} \|H_p\|_{L^p(\Omega)} \|\xi\|_{L^p(\Omega)} + \|h(\cdot, 0)\|_{L^p(\Omega)} \right)^{\frac{\eta}{p-\eta}} \right)^{\frac{1}{p-1}}, & \text{if } \tau = p,

\left( C \left( |\Omega|^{\frac{1}{p}} \|H_p\|_{L^p(\Omega)} \|\xi\|_{L^p(\Omega)} + \|h(\cdot, 0)\|_{L^p(\Omega)} \right)^{\frac{\eta}{p-\eta}} \right)^{\frac{1}{p-1}}, & \text{if } \tau \in (0, p),
\end{array}
\right.
\]

We obtain
\[
\|w\|_{L^p(\Omega)} \leq R.
\]

Since, according to Proposition 2.2 the set \(N(\overline{B}_R)\) is relatively compact in \(L^p(\Omega)\). Therefore, the compactness of \(N\) follows.

Thus, by Schauder fixed-point theorem, we conclude that \(N\) possesses a fixed point \(w \in W_0^{s,p}(\Omega)\) which is solution to problem (1.1).
Remark 1. Problem (1.1) has a nonnegative weak solution \( w \in W_0^{s,p}(\Omega) \), if (1.5)-(1.6) and one of

\[
\eta = 0, \delta = \frac{1 - p}{\tau} \quad \text{and} \quad \begin{cases} 
C |\Omega|^{\frac{1}{p}} \|H_p\|_{L^\infty(\Omega)}^{\frac{1}{p}} < 1, & \text{if } \tau = p, \\
C |\Omega|^{\frac{1}{p}} \|H_\tau\|_{L^\infty(\Omega)}^{\frac{1}{p}} < 1, & \text{if } \tau \in (0, p), 
\end{cases}
\]

\[
\eta \in \left(0, \frac{p - 1}{p}\right), \delta \leq 0, \delta = \frac{p - 1}{\tau} \quad \text{and} \quad \begin{cases} 
C |\Omega|^{\frac{1}{p}} \|\eta^{\frac{1}{p-1}}H_p\|_{L^\infty(\Omega)}^{\frac{1}{p}} < 1, & \text{if } \tau = p, \\
C |\Omega|^{\frac{1}{p}} \|\eta^{\frac{1}{p-1}}H_\tau\|_{L^\infty(\Omega)}^{\frac{1}{p}} < 1, & \text{if } \tau \in (0, p), 
\end{cases}
\]

\[
\eta \in \left[0, \frac{p - 1}{p}\right], \delta \leq 0, 0 < \eta - \delta < \frac{p - 1}{\tau} \quad \text{and} \tau \in (0, p),
\]

holds.

5 CONCLUSION

The paper provides a comprehensive discussion on the existence of nonnegative weak solutions to nonlocal quasilinear elliptic problem involving the fractional p-Laplacian operator. We have used Tychonoff and Schauder fixed-point theorems respectively, to get two existence theorems of weak positive solutions under some hypothesis on the data. As far as we are aware also, that the main results obtained above are new even in the semilinear.

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**REFERENCES**


