Novel 2-D cascade ladder-lattice structure recursive digital filters

Novos filtros digitais recursivos de estrutura de escada em cascata 2-D

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ABSTRACT

We present in this paper a novel cascade ladder-lattice structure of two-dimensional (2-D) recursive digital filters; this cascade based on two delay units M and N and a basic lattice section. Our new realization has a minimal number of basic lattice sections. Using a transfer function representation; the matrix representations for the basic lattice sections are proposed and used to derive the transfer functions of the realized 2-D digital lattice filters. The theory and transfer functions of the realized 2-D lattice Digital filter are derived. We use the 2-D Givone–Roesser’s state space model to verify the minimal realization of the proposed 2-D recursive digital filter; this later gives matrix-vectors A, b, c and a scalar d which are derived. As result, the dimension of the 2-D generalized state...
space model is minimal. Finally, several design examples are provided for conducting illustration.

Keywords: 2-D recursive digital filters, basic lattice section, lattice-ladder filters, Givone-Roesser’s state space.

RESUMO
Apresentamos neste artigo uma nova estrutura em cascata de filtros digitais recursivos bidimensionais (2-D); esta cascata é baseada na unidade de atraso M e N e em uma seção de rede básica. Nossa nova realização possui um número mínimo de seções básicas de rede usando uma representação de função de transferência; As representações matriciais para as seções básicas da rede são propostas e usadas para derivar as funções de transferência dos filtros de rede digital 2-D realizados. A teoria e as funções de transferência do filtro digital de rede 2-D realizado são derivadas. Usamos o modelo de espaço de estados 2-D de Givone-Roesser para verificar a realização mínima do filtro digital recursivo 2-D proposto; isso mais tarde fornece os vetores de matriz A, b, c e um escalar d que são derivados. Como resultado, a dimensão do modelo de espaço de estados generalizado 2-D é mínima. Finalmente, vários exemplos de design são fornecidos para conduzir ilustrações

Palavras-chave: filtros digitais recursivos 2-D, seção de rede básica, filtros de rede, espaço de estados de Givone-Roesser.

1 INTRODUCTION
In the last decade, the study of multidimensional systems for signal filtering has increasingly attracted the attention of many research teams; particular areas applications in it are; biomedicine, X-ray technology, video signal filtering, computational tomography, digital signal and image processing [1-4]. The problem of determining a state-space model representation for a given transfer function is important in system theory and because of the inherent absence of the non-applicability of the fundamental theorem of algebra to polynomials more than one variable, the crucial problem for multidimensional filters and systems [5]. Therefore, we require having digital filters with a minimum number of delay elements. This requirement has a mater not only by the material side specifications, but also because sometimes no minimal realizations often cause theoretical or computational difficulties. Since there is no entirely minimal circuit and state space realization, even in 2-D systems, except for special cases like the all-pole or all-zero filters, product factorable transfer functions, first order all-pass and lattice filters [6-7].
It is important to consider these structures even for low dimension filters and benefit of these structures by their applications [8].

This paper aims to propose a new lattice-ladder structure of 2-D digital filter, composed of a delay unit $z_1^{-1}$ and a basic lattice section $z_1z_2^{-1}$ other than the one proposed in [7-9], which are two delay units $z_1^{-1}$ and $z_2^{-1}$ are used. Furthermore, we have considered the proposed lattice structure by using a state-space 2-D Givone–Roesser’s form description presented in [9] and the implementations recommended by the authors in [10-14]. Using a matrix representation of the basic lattice sections and we derived transfer functions of the proposed 2-D digital ladder-lattice filters. The proposed structure with the minimal number of delays is composed of the minimal number of basic lattice sections. Besides, we present the state space equations for the 2-D digital filter, where the dimension of the matrix–vectors of the proposed cascaded ladder-lattice structure verifies the minimal state space realization.

We present the first–order 2-D recursive ladder-lattice digital filter with two basic lattice sections in Figure 1. To clarify this novel work; an interesting design example is investigated. The remainder of the paper is structural as follows. Section 2 is devoted to the proposed 2-D lattice-ladder structure realization. The state space realization is given in Section3. In the last section, a numerical example presented in section 4 and followed by conclusion and references.

2 PROPOSED LATTICE STRUCTURE AND REALIZATION

2.1 PROPOSED LADDER LATTICE 2-D DIGITAL FILTER

Figure 1 shows the proposed lattice –ladder structure to construct a discrete 2-D filter. This structure has a $z_1^{-1}$ delay units and $z_1z_2^{-1}$ basic lattice section.

The input-output relationship of the proposed lattice structure shown in Figure 1, is given by the following matrix representation:
Figure 1 – The proposed lattice – ladder 2-D digital filter with two basic lattice sections.

\[
\begin{bmatrix}
X_2(z_1, z_2) \\
U_2(z_1, z_2)
\end{bmatrix} =
\begin{bmatrix}
1 & k_2 z_1 z_2^{-1} \\
k_2 & z_2^{-1}
\end{bmatrix}
\begin{bmatrix}
1 & k_1 z_1^{-1} \\
k_1 & z_1^{-1}
\end{bmatrix}
\begin{bmatrix}
U_0(z_1, z_2)
\end{bmatrix}.
\]  

(1)

\[
Y(z_1, z_2) = X_2(z_1, z_2)v_2 + X_1(z_1, z_2)v_1 + X_0(z_1, z_2)v_0
\]

(2)

\[
U_0(z_1, z_2) = X_0(z_1, z_2).
\]

(3)

From (1), (2), (3) and after rearrangement, we can construct the transfer function of a 2-D recursive digital filter as follows:

\[
H_2(z_1, z_2) = \frac{Y(z_1, z_2)}{U_2(z_1, z_2)}
\]

\[
H_2(z_1, z_2) = \frac{v_0+v_1k_1v_2+k_2z_2^{-1}+(v_1+k_1k_2z_2^{-1})z_1^{-1}+v_2z_1^{-1}+k_1v_2z_1z_2^{-1}}{1+k_1z_1^{-1}+k_2z_2^{-1}+k_1k_2z_1z_2^{-1}}
\]

(4)

2.2 REALIZATION

To construct a general 2-D recursive lattice- ladder structure with order MxN, we generalize the structure given in Figure 1 by cascading N $z_1^{-1}$ and M $z_2^{-1}$ delay basics lattice section as shown in Figure 2.

The related state space 2-D model has the following structure:
\[
\dot{x}(i,j) = Ax(i,j) + bu(i,j)
\]

\[
y(i,j) = cx(i,j) + du(i,j)
\]

\[
x(i,j) = \begin{bmatrix}
    x_{M+N}(i,j) \\
x_{M+N-1}(i,j) \\
    \vdots \\
x_N(i,j) \\
x_{N-1}(i,j) \\
    \vdots \\
x_1(i,j)
\end{bmatrix},
\dot{x}(i,j) = \begin{bmatrix}
    x_{M+N}(i-1,j+1) \\
x_{M+N-1}(i-1,j+1) \\
    \vdots \\
x_{N+1}(i-1,j+1) \\
x_N(i+1,j) \\
    \vdots \\
x_1(i+1,j)
\end{bmatrix}
\]

(5)

Where:

- \(u(i,j)\): is the input;
- \(y(i,j)\) is the output and \(x(i,j)\) is the state;
- \(A\) is the state transition matrix.

The dimensions of the matrices \(A, b, c\) are \((M+N)\times(M+N), (M+N)\times1, 1\times(M+N)\), respectively.

The main issue of the minimal state space model is to form the matrices \(A, b,\) and \(c\) with the minimal dimensions.

In the following, we show that the proposed lattice-ladder structure composed of \((M+N)\) basic lattice sections produces the minimal dimensions of the matrices.

In the base of the \(z\)-transform for both sides of (5), we obtain the following transfer function:

\[
H_{M+N}(z_1, z_2) = \frac{Y(z_1, z_2)}{U_{M+N}(z_1, z_2)} = d + c(Z - A)^{-1}b,
\]

(6)

\[
Z = (z_2z_1^{-1}I_{M+N} \oplus \ldots \oplus z_2z_1^{-1}I_{M+N} \oplus z_1I_{M+N} \oplus z_1I_{M+N})
\]

With \(\oplus\) denoted the direct sum.

3 CIRCUIT AND STATE SPACE REALIZATION

The new 2-D ladder-lattice circuit generated as depicted in Figure 2; and in order to derive the state space model (5); it assumed that the outputs of the delay elements correspond to the states of the model. Moreover, by writing one state
equation for every delay element of the state space model and after some algebraic manipulations, we can conclude that the matrix–vectors $A$, $b$, $c$, and the scalar $d$, of the state space model, are derived by inspection. Not that the new 2-D structure has a minimal number of basic lattice section elements $z_1$, and also we noted that the cascaded circuit implementation with the minimal number of delays is composed of the minimal number of basic lattice sections $(M+N)$. The matrix–vectors $A$, $b$, $c$, and the scalar $d$ of the derived 2-D state space model of Roesser's type have the following form.

Figure 2 – The proposed lattice-ladder 2-D digital filter with order MXN

\[
\begin{align*}
A &= \begin{bmatrix}
-k_{M+N-1}k_{M+N} & 1 - k_{M+N-1}^2 & \cdots & 0 & 0 & 0 \\
-k_{M+N-2}k_{M+N} & -k_{M+N-2}k_{M+N-1} & 1 - k_{M+N-2}^2 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-k_{1}k_{M+N} & -k_{1}k_{M+N-1} & \cdots & -k_{1}k_{2} & 1 - k_{1}^2 \\
-k_{M+N} & -k_{M+N-1} & \cdots & -k_{2} & -k_{1}
\end{bmatrix} \\
b &= \begin{bmatrix}
k_{M+N-1} & k_{M+N-2} & \cdots & k_{1} & 1\end{bmatrix}^T \\
c &= \begin{bmatrix}
c_1 & c_2 & \cdots & c_{M+N-1} & c_{M+N}\end{bmatrix}
\end{align*}
\]
\begin{align*}
\mathbf{c}_i &= (1 - k_{M+N-1}^2)v_{M+N-1} - \sum_{i=1}^{M+N-1} k_i k_{M+N-1}v_i - k_{M+N-1}v_0, \quad i=j-1 \\
\mathbf{c}_{N+M-1} &= (1 - k_2^2)v_2 - k_1 k_2 v_1 - k_2 v_0, \\
\mathbf{c}_{N+M} &= (1 - k_1^2)v_1 - k_1 v_0, \\
d &= \sum_{i=1}^{M+N} k_i v_i + v_0. \quad (7)
\end{align*}

Noted that the realization describes minimum number of states \(1 \times (N+M)\), and the state space matrix \(A\) has minimal dimension \((N+M) \times (N+M)\), resulting from the corresponding minimal circuit realization.

\(k_i, \, i = 1 \ldots M+N.\) are reflection coefficients.

\(v_i, \, i = 1 \ldots M+N.\) are ladder network.

**4 EXAMPLE APPLICATION**

*4.1 FIRST ORDER 2-D LADDER-LATTICE DIGITAL FILTER*

To simplify the considered first order 2-D ladder-lattice filter, we take \(N=M=1\) and then we substituting into (5) and (7). The results corresponding the state space realization takes the form:

\begin{align*}
\dot{x}(i, j) &= A x(i, j) + b u(i, j), \\
y(i, j) &= c x(i, j) + d u(i, j),
\end{align*}

where:

\[
\begin{bmatrix}
\dot{x}_1(i, j) \\
\dot{x}_2(i, j)
\end{bmatrix} =
\begin{bmatrix}
x_1(i+1, j) \\
x_2(i, j)
\end{bmatrix},
\begin{bmatrix}
x_1(i, j)
\end{bmatrix} =
\begin{bmatrix}
x_2(i-1, j + 1) \\
x_1(i, j + 1)
\end{bmatrix}.
\quad (8)
\]

The matrix \(A\), the vectors \(b, \, c\) and the scalar \(d\) have the following quadruple state–space form:
$$
\begin{bmatrix}
[A] & [b]
\end{bmatrix},

\begin{bmatrix}
[c] & [d]
\end{bmatrix},

\text{or,}

\begin{bmatrix}
-k_1k_2 & 1 - k_1^2 \\
-k_2 & -k_1
\end{bmatrix}
\begin{bmatrix}
[k_1] \\
1
\end{bmatrix},

\begin{bmatrix}
(1 - k_2^2)v_2 - k_1k_2v_1 - k_2v_0 \\
(1 - k_1^2)v_1 - k_1v_0
\end{bmatrix}
\begin{bmatrix}
a_1
\end{bmatrix},

a_1 = v_0 + k_1v_1 + k_2v_2.

The dimension of the state space \(x(i,j)\) is 1x2.

The dimensions of the matrix–vectors \(A, b, c\) are 2x2, 2x1, 1x2, respectively.

The state space is minimal

The 2-D transfer function of the state space by using (6) and (9) is:

\[H_2(z_1, z_2) = v_0 + k_1v_1 + k_2v_2 + [(1 - k_2^2)v_2 - k_1k_2v_1 - k_2v_0 \ (1 - k_1^2)v_1 - k_1v_0]

\begin{bmatrix}
[z_2z_1^{-1} & 0] \\
0 & z_1
\end{bmatrix}
\begin{bmatrix}
-k_1k_2 & 1 - k_1^2 \\
-k_2 & -k_1
\end{bmatrix}^{-1}
\begin{bmatrix}
[k_1] \\
1
\end{bmatrix},

\[H_2(z_1, z_2) = \frac{v_0 + v_1k_1 + v_2k_2 + [(v_1 + k_1k_2v_2)z_1^{-1} + v_2z_2^{-1} + k_1v_2z_1z_2^{-1}]}{1 + k_1z_1^{-1} + k_2z_2^{-1} + k_1k_2z_1z_2^{-1}}\]

(10)

We can see that the transfer function (10) is the same as (4) and the minimal number of delay units is equals to the minimal number of basic lattice sections.

For \(v_2 = 1\) and \(v_1 = v_0 = 0\), the above transfer function (10) takes the form:

\[H_2(z_1, z_2) = \frac{k_2 + k_1k_2z_1^{-1} + z_2^{-1} + k_1z_1z_2^{-1}}{1 + k_1z_1^{-1} + k_2z_2^{-1} + k_1k_2z_1z_2^{-1}}\]

(11)

\[H_2(z_1, z_2) = z_2^{-1} \frac{k_2 + k_1k_2z_1^{-1} + z_2^{-1} + k_1z_1z_2^{-1}}{1 + k_1z_1^{-1} + k_2z_2^{-1} + k_1k_2z_1z_2^{-1}}.\]

(12)

\[H_2(z_1, z_2) = z_2^{-1} \frac{D_2(z_1^{-1}, z_2^{-1})}{D_2(z_1, z_2)}\]

(13)
It is obvious that the above transfer function (13) can provide filters having all–pass and all–pole characteristics as in [11].

Notice in particular that, for the ladder network \( v_{M+N} = 1 \) and \( v_i = 0, \ i = 0, 1, M+N-1 \), we find the proposed structure in [11].

### 4.2 2-D LADDER-LATTICE FILTER WITH LOW DIMENSION FILTER

In this example we consider a low dimension filter \( N=2 \) and \( M=1 \), a 2-D ladder-lattice filter as shown in Figure 3. The corresponding state space realization takes the form:

\[
x(i, j) = \begin{bmatrix} x_3(i, j) \\ x_2(i, j) \\ x_1(i, j) \end{bmatrix}, \quad x(i, j) = \begin{bmatrix} x_3(i - 1, j + 1) \\ x_2(i + 1, j) \\ x_1(i + 1, j) \end{bmatrix}
\]  

(14)

The quadruple matrices form for this example is:

\[
\begin{bmatrix}
-k_3 k_2 & 1 - k_2^2 & 0 \\
-k_1 k_3 & -k_2 & 1 - k_1^2 \\
-k_3 & -k_2 & -k_1 \\
\end{bmatrix} \begin{bmatrix}
k_2 \\
k_1 \\
1 \\
\end{bmatrix}
\]

\[
(1 - k_3^2)v_3 - k_1 k_3 v_1 - k_2 k_3 v_2 - k_3 v_0 \\
(1 - k_2^2)v_2 - k_1 k_2 v_1 - k_2 v_0 \\
(1 - k_2^2)v_2 - k_1 k_2 v_1 - k_2 v_0 \\
\]

(15)

Figure 3 – Lattice- ladder 2-D digital filter with \( N=2 \) and \( M=1 \)

Source: Authors.

The dimension of the state space \( x(i, j) \) is 1x3.
The dimensions of the matrix–vectors $A$, $b$, $c$ are 3x3, 3x1 and 1x3, respectively.

4.3 NUMERICAL EXAMPLE

A numerical example is performed the lattice-ladder 2-D digital filter Figure 3 and by using (6) and (15) to design of 2-D transfer function $H_3(z_1,z_2)$ as follows in (16)

Where:

$$a_1 = v_0 + v_1 k_1 + v_2 k_2 + v_3 k_3$$ \hspace{1cm}(16)$$

Hence from the transfer function (16), we can find some results, such as:

When we substitute the values of the ladder network $v_0 = v_1 = v_2 = 0$ and $v_3 = 1$ in (16), we get a magnitude response has a unity gain at all frequencies $(\omega_1, \omega_2)$, we have the 2-D all pass filter.

When we substitute the values of the ladder network $v_1 = v_2 = v_3 = 0$ in (16), we get a 2-D all pole filter in (16).

In this section, we establish a 2-D parallel line filter using the 2-D all pole filter (16) which is a bandstop filter with a two very narrow passband and three stopbands. The 2-D parallel line filters are characterized by The cut off frequency of the parallel line filter $\omega_p$ and a 3-dB pass Bandwidth (BW).

The frequency specification of 2-D parallel line filter is given [13]:

$$H_p(e^{j\omega_1}, e^{j\omega_2}) = \begin{cases} 1 & \text{if } \omega_2 = \pm \omega_p, \\ 0 & \text{if otherwise}. \end{cases}$$ \hspace{1cm}(17)$$

The implementation of the 2-D parallel line filter takes the procedure as follows.

After designing the transfer function in (17).

Specify the cutoff frequency of the parallel line filter $\omega_p$ which is related to the coefficient $k_1$ in[14-17] by:

$$k_1 = - \cos(\omega_p).$$ \hspace{1cm}(18)$$
The 3-dB pass bandwidth (BW) is related to the coefficient \( k_2 \) in Figure 4 with

\[
k_2 = \frac{1-\tan\left(\frac{\text{BW}}{2}\right)}{1+\tan\left(\frac{\text{BW}}{2}\right)}.
\]

To satisfy that the projection on \((\omega_2, \|H_p(e^{j\omega_1}, e^{j\omega_2})\|)\) plan (18) and \(\omega_2 = \omega_p\) for all \(\omega_1\), the coefficient \( k_3 \) is related to the coefficient \( k_2 \) by

\[
k_3 = 1 - k_2.
\]

To have the unite gain at the amplitude, we must take

\[
v_0 = \frac{1}{\max\|H_p(e^{j\omega_1}, e^{j\omega_2})\|},
\]

4.3.1 Design of 2-D Parallel Line Filter

Consider the design of 2-D parallel line filter with \(\omega_p = 0.7\pi, \text{BW} = 0.001\pi\), then we follow the steps discussed previously.

From (18), \(k_1 = -\cos(0.7\pi) = 0.5878\).

From (19) and (20), \(k_2 = 0.9969, \ k_3 = 0.0031\).

From (21), \(v_0 = 2.5297 \times 10^{-3}\).

For an illustration, by substituting the coefficients \(k_1, k_2, k_3\) and \(v_0\) in (17), the magnitude response is consequently obtained and shown in Figure 5 (a, b).

In Figure 5- (a): the 2-D parallel line filter is presented where the cutoff frequency is localized at \(\omega_p = \pm 0.7\pi\).

The projection on \((\omega_2, \|H_p(e^{j\omega_1}, e^{j\omega_2})\|)\) plan is presented in Fig. 5- (b).

We can take another example with: \(\omega_p = 0.2\pi\) and \(\text{BW} = 0.003\pi, \ k_1 = -0.8090, k_2 = 0.9906, k_3 = 0.0094, v_0 = 5.4621 \times 10^{-3}\).

The magnitude response of this filter is shown in Figure 5 (c, d).

The cutoff frequency is operated at \(\omega_p = \pm 0.2\pi\).

In addition, we can illustrate the 2-D parallel line filter in the direction of \((\omega_1, \|H_p(e^{j\omega_1}, e^{j\omega_2})\|)\), by using the transformation \((H_p(z_1,z_2)\) to \(H_p(z_2,z_1))\).
\[ H_3(z_1, z_2) = \frac{a_1 + (v_1 + k_1 v_2 (1 + k_2) + \nu_3 k_1 k_3 (1 + k_2)) z_1 z_2^{-1} + k_1 (1 + k_2) v_3 z_2^{-1} + k_2 v_3 z_1 z_2^{-1} + (v_2 + k_2 k_3 v_3) z_1^2 + v_3 z_1^{-1} z_2^{-1}}{1 + k_1 (1 + k_2) z_1^{-1} + k_4 k_3 (1 + k_2) z_2^{-1} + k_3 k_3 k_2 z_1 z_2^{-1} + k_2 z_1^2 + k_3 z_1^{-1} z_2^{-1}} \]  

(22)

\[ H_p(z_1, z_2) = \frac{v_0}{1 + k_1 (1 + k_2) z_1^{-1} + k_4 k_3 (1 + k_2) z_2^{-1} + k_3 k_3 k_2 z_1 z_2^{-1} + k_2 z_1^2 + k_3 z_1^{-1} z_2^{-1}} \]  

(23)

Figure 4. Variation filter bandwidth with variation of \( k_2 \), for \( \omega_p = \pm 0.5\pi \).
5 CONCLUSION

This paper proposed a new ladder-lattice structure of 2-D recursive digital filters. The transfer function calculated by using the matrix representations and derived the corresponding state-space realizations of a cascaded lattice-ladder 2-D digital structured filter. The minimal number of delay units is equal to the minimal number of 2-D basic lattice sections. The verification of the characteristics of the constructed 2-D recursive digital filter by the Roesser’s 2-D state-space model confirms our innovative work; noted this circuit configuration characterized by a minimal realization. The development and examples show that the low dimension 2-D ladder-lattice filter can design 2-D parallel line filters. This studied allow us to use these filters for specific purposes like notch filters.

In perspective we hope continue our work to extended this study to the 3-d n-d digital lattice filters.

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